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A Probabilistic Model for Run-Length Coding of Pictures

A Note on Invariant Relations for Ambiguity and Distance Functions

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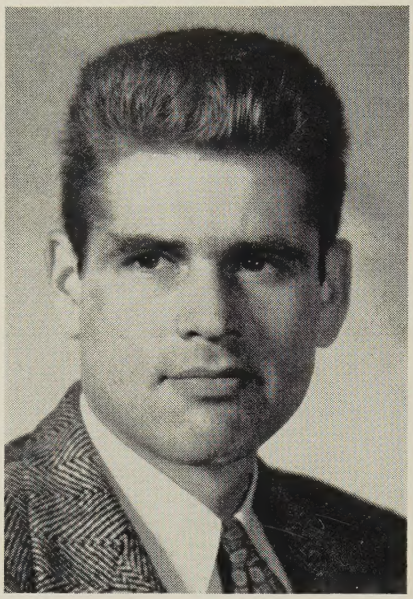
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Peter Elias

Peter Elias (S '48—A '51—M '56) was born on November 26, 1923, in New Brunswick, N. J. He received the S.B. degree from the Massachusetts Institute of Technology, Cambridge, in 1944, and the M.A., M.E.S., and Ph.D. degrees in 1948, 1949, and 1950, respectively, from Harvard University. He served with the U. S. Navy as a radio technician from 1944 to 1946.

From 1950 to 1953, he was a Junior Fellow in the Society of Fellows at Harvard University, doing research on information theory. He became an assistant professor in the Electrical Engineering Department and the Research Laboratory of Electronics at M.I.T. in 1953. In 1956 he became an associate professor.

He has been working on two-dimensional random processes and pictures as information sources, efficient coding of redundant sources, and noisy channel coding problems.

Dr Elias is a member of U. S. A. Commisions 6.1 and 6.2 of URSI, Sigma Xi, Eta Kappa Nu, and the Institute of Mathematical Statistics. He is the coeditor, with C. Cherry and L. Brillouin, of "Information and Control."

PGIT in 1960

1960 should be an interesting year for PGIT. Eleven years after the publication of C. E. Shannon's basic work, the first serious attempts to apply coding to get reliable transmission of digital data over unreliable channels have started in several laboratories.

Interest in finding engineering solutions for the problems involved will undoubtedly spread further during this year. Sending reliable information around the solar system using limited power is one of the applications which is causing increased interest. We shall probably be seeing reports on progress in the implementation of coding and decoding techniques in coming issues of these TRANSACTIONS.

After much discussion, PGIT is introducing two new kinds of special issues of TRANSACTIONS. One is the monograph—too long for publication as a journal article, too specialized for publication as a book—which can usually be distributed only in the form of a technical report to those lucky enough to have heard of it. The first issue of this sort, a well-known but unpublished report by J. I. Marcum on radar detection problems, is in the mill. So is the first number of another type of special issue: a series of submitted papers on matched filter techniques, being organized by Paul E. Green, Jr., Vice-Chairman of PGIT and Associate Editor of these TRANSACTIONS. Other special issues will be coming along, and suggestions along these lines should be directed to P. E. Green, Jr. at M.I.T. Lincoln Laboratory, Lexington, Mass.

In addition to publishing TRANSACTIONS, PGIT organizes meetings, or cooperates with others who do so. The success of the International Symposium on Circuit and Information Theory in Los Angeles last June, organized by G. L. Turin, R. A. Epstein, and others, has suggested that we should program such meetings regularly in the United States every two years. Such programming would simplify the job of the organizers of the meetings, since there would be longer notice and more time to get the many necessary jobs done. In the intervening years we might cosponsor or even help to organize meetings held in other countries. Along these lines we are beginning to plan for a meeting on the East Coast during the Spring or Fall of 1961.

I look forward to hearing any suggestions you may have for improving the effectiveness of PGIT activity in connection with meetings, publications or any other services which the group might effectively perform.

—PETER ELIAS

A Class of Systematic Codes for Non-Independent Errors*

N. M. ABRAMSON†

Summary—**A class of systematic codes has been obtained which will correct all single errors and all double errors which occur in adjacent digits. These codes use significantly fewer checking digits than codes which correct *all* double errors. In addition, because of inherent regularities in their structure, these codes may be instrumented in a strikingly simple fashion.**

Introduction

THE occurrence of independent errors in a binary communication system may be combatted quite effectively by the use of single and multiple error-correcting block codes [1]–[4]. In many cases, however (*e.g.*, telephone lines with impulse noise and magnetic tape with signal "drop-out"), errors do not occur independently; the probability that the nth binary digit will be in error will depend upon whether the $(n-1)$th binary digit was in error. That is, in many situations, if two errors occur in one word, the errors are probably in adjacent binary digits. Using a double-error correcting code in such situations will increase the reliability of transmission at the cost of greatly increasing the number of binary digits necessary to transmit each word. Such a code clearly does not make use of the fact that double errors are likely to be adjacent errors.

This paper discusses the construction of a class of systematic codes[1] which corrects all single errors and all double errors when these double errors occur in adjacent binary digits. These codes require significantly fewer parity check digits than codes which correct all double errors. Furthermore, because of inherent regularities in the structure of these codes, they can be instrumented with remarkable simplicity.

We have organized this paper into three sections. Section I presents a set of rules without their derivation, which will allow the reader to construct single-error-correcting-double-adjacent-error-correcting (SEC-DAEC) codes. A simple discussion indicating how the codes might be instrumented is also included in Section I. Section II contains a detailed example of the construction and use of a SEC-DAEC code. Section III contains a discussion of the properties of SEC-DAEC codes in general, and defines the subclass of SEC-DAEC codes obtained in this paper. Section III also contains a derivation of the rules presented in Section I, and some miscellaneous observations on the properties of the codes.

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[1] See sec. 7 of Hamming [1].

Section I

A. The Number of Information Digits

Because we restrict our attention to systematic block codes, we may write any possible word of our code as

$$\overbrace{\underbrace{x_1 x_2 \cdots x_m}_{m \text{ binary digits}} \; \underbrace{y_1 y_2 \cdots y_k}_{k \text{ binary digits}}}^{n \text{ binary digits}}. \tag{1}$$

That is, we shall assume that each word is n binary digits long. Of these n digits, m digits may be used to convey the information; we denote these by $x_1, x_2, \cdots x_m$. The remaining k digits, denoted by $y_1, y_2, \cdots y_k$, are determined by suitable parity checks over the information digits.

The first property of the codes which must be determined is the number of parity digits necessary for any given number of information digits. Stated another way, we might ask for the largest number of information digits we may use for a specified number of parity digits, k. In Table I, we have listed, for $k = 1, 2, \cdots 10$, the maximum number of information digits possible for the class of SEC-DAEC codes obtained. This upper bound is denoted by m^*. For purposes of comparison, we have also given in the same table, m^{**}, an upper bound on the number of information digits possible for codes which correct *all* double errors.

We shall present a set of rules for constructing SEC-DAEC codes where, for any given number of parity digits k, the number of information digits is m^*, the maximum possible as indicated in Table I. Codes with the number of information digits less than m^* may be constructed by first obtaining the code for m^* information digits in the usual manner, and then setting the first few information digits equal to zero.

B. The Parity Check Table

To specify a systematic code completely, it is only necessary to specify the digits checked by each of the parity digits. For example, in a SEC-DAEC code of total block length 7, where we have 3 information digits and 4 parity digits, we can completely specify a code by filling in a parity check table, as shown in Fig. 1.

This is merely a convenient way of indicating that for this code

y_1 checks x_1 and x_2
y_2 checks x_2 and x_3
y_3 checks x_3 and y_1
y_4 checks all the digits.

TABLE I
THE NUMBER OF INFORMATION DIGITS FOR SEC-DAEC AND SEC-DEC CODES

k	m^*	m^{**}
1	—	—
2	—	—
3	0	—
4	3	1
5	10	2
6	25	4
7	56	8
8	119	14
9	246	22
10	501	34

k = number of parity digits.
m^* = upper bound on the number of possible information digits for the class of SEC-DAEC codes obtained.
m^{**} = upper bound on the number of possible information digits for ordinary double error correcting codes.

TABLE II
PARITY CHECKS FOR y_1 IN SEC-DAEC CODES (M SEQUENCES)

Digit Number	$k = 4$	$k = 5$	$k = 6$	Digit Number	$k = 6$
1	0	0	0	17	0
2	0	1	0	18	1
3	1	0	0	19	0
4	0	1	1	20	0
5	1	1	1	21	1
6	1	0	0	22	0
7	1	0	1	23	1
8		1	1	24	1
9		0	1	25	0
10		0	0	26	0
11		0	1	27	1
12		1	0	28	1
13		1	1	29	1
14		1	0	30	1
15		1	0	31	1
16			0		

	x_1	x_2	x_3	y_1	y_2	y_3	y_4
y_1	x	x		x			
y_2		x	x		x		
y_3			x	x		x	
y_4	x	x	x	x	x	x	x

Fig. 1—A parity check table.

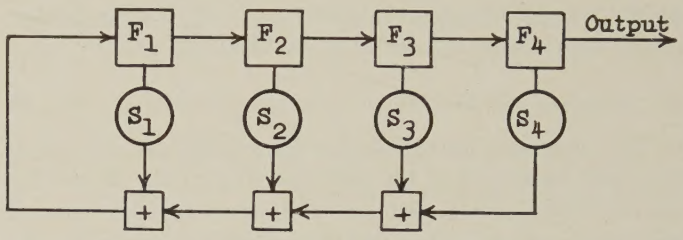


Fig. 2—A four-stage shift register.

The rules, then, for filling out a parity check table of rows (corresponding to the k parity digits) and $m^* + k = n$ columns (corresponding to the m^* information digits and the k parity digits) are:

1) Number the rows $y_1, y_2, \cdots y_k$, as indicated in the example above.
2) Let y_k be a parity check over all the digits.
3) The parity checks for y_1 may be determined from Table II. For any given k, use the sequence of zeros and ones in Table II to obtain the parity checks on y_1 as follows:

   For any k the corresponding sequence in Table II is $m^* + k$ digits long. If the jth digit in this sequence is a zero, y_1 will check the jth digit of the code words; if the jth digit in this sequence is a one, y_1 will not check the jth digit of the code words. That is, we need only write the sequence obtained from Table II in the first row of the parity check table we are constructing, using x for a zero and a blank space for a one. For example: for $k = 4$, Table II gives the sequence

   0 0 1 0 1 1 1

   and in the parity check table shown in Fig. 1, the first row has x in the first, second, and fourth places.

4) The parity checks in each succeeding row of the parity check table (except the last, y_k) are then the same as the row directly above, except that they are shifted to the right by one digit (see Fig. 1).

These four rules, together with Table II, allow one to construct SEC-DAEC codes of block length less than, or equal to, 31 digits. It is, of course, possible to construct SEC-DAEC codes with arbitrary block length. This construction is explained directly below.

C. Shift Registers

The fact that the codes obtained in this paper may be simply instrumented depends directly upon the properties of the sequence of zeros and ones giving the parity checks for y_1 (see Rule 3 above). Each of the sequences of length $2^R - 1$, listed in Table II, may be derived from an R-stage binary shift register. Fig. 2 is a block diagram of a four-stage register. (For the most part, we follow the notation of Zierler [5].)

The F_i are flip-flops, the circles are switches ($S_i = 0$ means open, $S_i = 1$ closed), and the small squares are mod-2 adders. For a given set of zeros and ones, S_1, S_2, S_3, and S_4, we insert an arbitrary binary number (except 0000) into the flip-flops. This binary number is then shifted to the right every T seconds, while simultaneously we feed into F_1, as indicated in Fig. 2. The successive entries of zeros and ones into F_1 then form a linear binary

shift-register sequence. There is extensive literature on the properties and uses of these sequences [4]–[7].

For certain values of the S_i, the successive entries in the flip-flops of an R-stage register will be all of the R-digit binary numbers, except the all-zero number. In this case, the sequence of zeros and ones in F_i is periodic of period $2^R - 1$, and is called a maximal length linear binary shift-register sequence, or m sequence [4]. *The sequence of zeros and ones used in Rule* 3 *can always be taken to be an m sequence out of a* $k - 1$*-stage shift register ending in* $k - 1$ *ones.*

Values[2] for the S_i which yield m sequences for R-stage registers, $R = 3, 4, \cdots 19$, are given in Table III. For example, for $R = 4$, we see that $S_1 = 1$, $S_2 = 0$, $S_3 = 0$, $S_4 = 1$. Putting all ones into the four flip-flops of Fig. 2 with these S_i will cause the sequence of length $2^4 - 1 = 15$ given in Table II to appear in F_1, starting with the first shift.

Using Table III, we may construct SEC-DAEC codes of block lengths from five digits to $2^{19} - 1$ digits, a range which seems adequate for most applications.

TABLE III

VALUES OF S_i FOR R-STAGE MAXIMAL LENGTH LINEAR BINARY SHIFT REGISTERS (SEE FIG. 2)

R	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
S_1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_3	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S_4		1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
S_5			1	1	0	1	1	0	0	0	0	0	0	0	0	0	0
S_6				1	1	1	0	0	0	1	0	0	0	0	0	0	0
S_7					1	0	0	1	0	0	0	0	0	0	0	0	0
S_8						1	0	0	0	1	0	0	0	0	0	0	0
S_9							1	0	1	0	1	1	0	0	0	0	0
S_{10}								1	0	0	1	0	0	0	0	0	0
S_{11}									1	1	0	0	0	1	0	0	0
S_{12}										1	1	0	0	0	0	0	0
S_{13}											1	0	0	1	0	0	0
S_{14}												1	1	0	1	0	1
S_{15}													1	1	0	1	0
S_{16}														1	0	0	0
S_{17}															1	0	1
S_{18}																1	1
S_{19}																	1

D. Encoding and Decoding

The encoding operation for these SEC-DAEC codes then can make use of the fact that the parity checks are derived from m sequences. For example, the timing signals for y_1 may be obtained directly from the first flip-flop of a $k - 1$-stage shift register, which is started with ones in all $k - 1$ flip-flops. The timing signals for y_2 to y_{k-1} are the same, but are shifted in time.

Decoding is almost as simple as encoding. The first step is to form the checking number $C_1C_2 \cdots C_k$ as in an ordinary Hamming code. That is, we set $C_i = 0$ if y_i of the received word satisfies its parity check; we set $C_i = 1$ if y_i does not satisfy its parity check. The timing signals for this operation are derived in the same manner as in the encoding operation.

There are, then, four possibilities:

1) All the C_i are zero: *No errors occurred.*
2) C_k is one; C_i is zero for some i: *A single error occurred.* We take a $k - 1$-stage shift register with the S_i given by Table III. This shift register is also started with all ones in the flip-flops. We start shifting the contents of the flip-flops in the usual manner. After some number of shifts, say N, the mod-2 sum of the number in the flip-flops and the first $k - 1$ digits of the checking number will all be ones.
   The error occurred in the Nth digit of the word.
3) C_k is zero; C_i is one for some i: *A double adjacent error occurred.*[3] We take a $k - 1$-stage shift register with the S_i given by Table III. This shift register (which can be the same register used for possibility 2), is started with a one in F_1, and zeros in all the other flip-flops. After N shifts, the mod-2 sum of the number in the flip-flops and the first $k - 1$ digits of the checking number will all be zeros.
   The double adjacent error occurred in the Nth and $N + 1$*th (mod-n) digits of the word.*
4) All the C_i are one: *An odd number of errors greater than one occurred.*

SECTION II

A. Formation of a Code

Let us assume we wish to obtain an SEC-DAEC code with ten information digits. From Table I, we see that it is necessary to use at least five parity digits. Using the notation of (1), then, we have

$$m = 10$$

$$k = 5$$

$$n = m + k = 15.$$

Now we construct a parity-check table of five rows and fifteen columns, as in Fig. 3.

The y_1 row was obtained from the sequence corresponding to $k = 5$ in Table II. The y_2, y_3, and y_4 rows are just shifted versions of the y_1 row. The y_5 row is, of course, trivial, since y_5 checks all digits.

B. Error Correcting

To demonstrate the error correcting process, we list in Table IV successive contents of the $k - 1$ or 4-stage shift register, starting with 1111 and also starting with 1000. From Table III, we see that $S_1 = S_4 = 1$, and $S_2 = S_3 = 0$.

Now assume that we receive a word where only the sixth digit is in error. From the parity check table of this code (Fig. 3), we see that the checking number corre-

[2] In general, these are not the only acceptable values for the C_i. This table was constructed with the aid of Marsh's Table [8].

[3] These codes consider the case where both the first and last digits of a word are in error, as a double adjacent error, and automatically correct the word, if this occurs.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	y_1	y_2	y_3	y_4	y_5
y_1	x		x			x	x		x	x	x				
y_2		x		x			x	x		x	x	x			
y_3			x		x			x	x		x	x	x		
y_4				x		x			x	x		x	x	x	
y_5	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

Fig. 3—Parity-check table for $m = 10$, $k = 5$.

TABLE IV
THE CONTENTS OF A FOUR-STAGE SHIFT REGISTER

Number of Shifts	(1111 Start)				(1000 Start)			
	F_1	F_2	F_3	F_4	F_1	F_2	F_3	F_4
1	0	1	1	1	1	1	0	0
2	1	0	1	1	1	1	1	0
3	0	1	0	1	1	1	1	1
4	1	0	1	0	0	1	1	1
5	1	1	0	1	1	0	1	1
6	0	1	1	0	0	1	0	1
7	0	0	1	1	1	0	1	0
8	1	0	0	1	1	1	0	1
9	0	1	0	0	0	1	1	0
10	0	0	1	0	0	0	1	1
11	0	0	0	1	1	0	0	1
12	1	0	0	0	0	1	0	0
13	1	1	0	0	0	0	1	0
14	1	1	1	0	0	0	0	1
$2^4 - 1 = 15$	1	1	1	1	1	0	0	0

sponding to an error in the sixth place is 10011. The last digit of the checking number is 1. Therefore, we start our shift register with 1111. After *6 shifts*, (see Table IV), the number in the flip-flops is 0110. This will give 1111 when added to the first four digits of the checking number, so we know that the error occurred in the sixth digit.

Let us now demonstrate the correction of a double adjacent error. Assume that the eighth and ninth digits of the received word are in error. From Fig. 3, we see that the checking number obtained will be 11010. The last digit of the checking number is zero. Therefore, we start our shift register with 1000.

After *eight shifts* (see Table IV), the number in the flip-flops is 1101. This will give 0000 when added to the first four digits of the checking number, so we know that the errors occurred in the eighth and ninth digits.

SECTION III

A. Complete SEC-DAEC Codes

In this section we shall first investigate the properties of SEC-DAEC codes in general. After this, we shall place some restrictions suggested by practical considerations upon the codes we wish to consider. Finally, we shall show how these restrictions lead to the rules given in Section I.

Consider the set of all possible checking numbers for a code with m information digits and k parity digits. There are 2^k of these; for an SEC-DAEC code, each of these 2^k numbers must indicate one of $2(m + k) + 1$ possibilities.[4] Therefore, we have the inequality

$$2^k \geq 2(m + k) + 1. \tag{2a}$$

Now the left side of (2a) is an even number, while the right side is an odd number, so that we may add one to the right side of (2a) without destroying the inequality.

$$2^k \geq 2(m + k) + 2 \tag{2b}$$

or, finally,

$$m \leq 2^{k-1} - k - 1. \tag{2c}$$

For any given k, let m^* be the value of m obtained by using the equality in the above equation. That is,

$$m^* = 2^{k-1} - k - 1. \tag{3}$$

We shall call an SEC-DAEC code with $m = m^*$ a *complete* SEC-DAEC code. Values of m^* may be obtained from from Table I.

B. The Checking Numbers

In order to specify completely an SEC-DAEC code of $n = m + k$ digits, it is sufficient to give the n checking numbers corresponding to the n possible single errors. Given these checking numbers, we can construct the parity check table as in Fig. 3, and thus the code itself. Let us therefore investigate the restrictions we must impose upon these checking numbers.

Let M_i be the checking number which indicates an error in the ith digit of the received word. Let N_i be the checking number which indicates an error in both the ith and $(i + 1)$th digits of the received word. Let I_k be the checking number which indicates that the received word contains no errors. I_k must, of course, be the k-digit binary number consisting of all zeros. Then we form an array of these k-digit binary numbers as shown below:

$$\begin{array}{lll} I_k & M_1 & N_1 \\ & M_2 & N_2 \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & M_n & N_n \end{array} \tag{4}$$

Now we note the following facts:

a) The M_i, N_i and I_k are all distinct.

b) Let $A = (a_1, a_2, \cdots a_k)$, and $B = (b_1, b_2, \cdots b_k)$ be two k-digit binary numbers.

[4] There are $(m + k)$ possible single errors, $(m + k)$ possible double adjacent errors (since an error in the first and last digits of the block is considered a double adjacent error) and the possibility of no errors.

We define the sum of A and B as follows:

$$(c_1, c_2, \cdots c_k) = C = A + B \tag{5}$$

where

$$c_i = a_i + b_i \qquad (\text{mod } 2).$$

Then we must have[5]

$$M_i + M_{i+1} = N_i \qquad i = 1, 2, \cdots n. \tag{6}$$

To these two facts, which must hold for any SEC-DAEC code, we may add a third, which is true for complete SEC-DAEC codes. For this type of code, we have [from (3)]

$$n = m^* + k = 2^{k-1} - 1 \tag{7}$$

so that we may state the third fact:

c) For a *complete* SEC-DAEC code, the $2n + 1$ checking numbers in the array (4) include *all* the k-digit binary numbers, except one.

The problem of finding a complete SEC-DAEC code with k parity-check digits, then, is equivalent to the problem of finding an array of k-digit binary numbers satisfying facts a)-c).

In Section III-D, we shall show how to construct such an array. Before doing so, however, it will be useful to impose an additional restriction on the array.

C. All-Check Codes

It is necessary that the parity checks display some sort of regularity so that the equipment to instrument a SEC-DAEC code does not make the use of such a code impractical. One restriction on the parity check digits which arises quite naturally from a consideration of the instrumentation of the code, is that there be a simply obtained distinction between a single error and a double adjacent error. The obvious way to obtain such a distinction is to require that some parity check digit, say y_k, be a parity check over all the digits. If y_k checks all the digits, then the digit of the check word corresponding to y_k will be 1 for a single error, and 0 for a double adjacent error (or no error). We therefore set the additional requirement on our codes that:

> The last parity digit, y_k, is a parity check over all the digits of the code word. (9)

We shall call a SEC-DAEC code obeying (9) an *all-check SEC-DAEC code.*

D. Properties of All-Check Codes

From this point on, then, we shall restrict our attention to all-check SEC-DAEC codes. It will simplify the discussion to follow, if we first show how the three facts listed in Section III-B are modified for all-check SEC-DAEC codes. Note that in the array of checking numbers (4) for such codes, the last digit of each M_i is a one, and the last digit of each N_i is a zero. We define A_i as the $k - 1$-digit binary number consisting of the first $k - 1$ digits of M_i, and B_i as the first $k - 1$ digits of N_i. Then, instead of an array of k-digit binary numbers, as in (4), we may consider the array of $k - 1$ digit binary numbers given below:

$$\begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_n & B_n \end{array} \tag{10}$$

Given the array (10) it is, of course, trivial to construct the array given in (4), and thus the corresponding SEC-DAEC code. Let us then express facts a)-c) in terms of the A_i and B_i, instead of the M_i and N_i. This will give us the three equivalent conditions [note the definition of addition in (5)]:

a′) $A_i + A_j = I_{k-1}$ if and only if, $i = j$
$B_i + B_j = I_{k-1}$ if and only if, $i = j$
$B_i \neq I_{k-1}$ for any i.

b′) $A_i + A_{i+1} = B_i$ $i = 1, 2, \cdots n$.

c′) For a *complete* all-check SEC-DAEC code, $(A_1, A_2, \cdots A_n)$ includes all the $k - 1$-digit binary numbers, except one, and $(B_1, B_2, \cdots B_n)$ also includes all the $k - 1$-digit binary numbers, except one, (I_{k-1}).

We shall call an array such as (10), satisfying the above three conditions, a *check array.*

E. The Check Array

If we are given one check array, it is possible to obtain other check arrays through certain simple operations. For example, if (10) is a check array, any cyclic permutation of the elements of (10) is also a check array. That is,

$$\begin{array}{cc} A_n & B_n \\ A_1 & B_1 \\ A_2 & B_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_{n-1} & B_{n-1} \end{array} \tag{11}$$

is a check array.

Furthermore, we may add an arbitrary $k - 1$ digit binary number, C, to each of the A_i in a check array, as in (12), directly below, and the result will still be a check array.

[5] Subscripts in this and following equations should always be taken modulo n.

$$\begin{array}{ll} A_1 + C & B_1 \\ A_2 + C & B_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_n + C & B_n \end{array} \qquad (12)$$

Note that the same is not true for the B_i.

By condition c') we know that the A_i and B_i each include all $k - 1$ digit binary numbers, except one. The number missing from the B_i must be I_{k-1}, while the number not included in the A_i we shall denote by A^*. By use of the operation indicated in (12), it is possible to set A^* of a check array equal to I_{k-1}. We shall call such a check array a *symmetric check array.*

F. Construction of the Codes

In Section III-E, we reduced the problem of constructing systematic SEC-DAEC codes to that of constructing symmetric check arrays. Now we shall present a constructive proof to show that a symmetric check array is possible for any $k \geq 3$. (The $k = 3$ check array, however, does not correspond to a SEC-DAEC code.)

The construction is quite simple. We let A_1 be an arbitrary $k - 1$-digit binary number. We start a $k - 1$-stage maximal-length linear binary shift register [4]–[7] with A_1, and let $A_2, A_3, \cdots A_n$ be the successive entries in the shift register. Given the A_i's, the B_i's are determined by b'). It is then only necessary to show that the set of A_i's and the set of B_i's each include all the $k - 1$-digit binary numbers, except I_{k-1}, once and only once. That this is true for the A_i's follows directly from the definition of a maximal-length linear binary shift register (see Section I-C). As for the B_i's, we show in the Appendix that the B_i's are just a shifted version of the A_i's. That is,

$$B_i = A_{i+s} \qquad (13)$$

for all i
and some fixed s ($i + s$ is taken mod n).

Therefore, the B_i's also include each $k - 1$-digit binary number, except I_{k-1}, once and only once, and the proof that this construction leads to a symmetric check array is complete.

G. A Modification

The previous section shows how to construct symmetric check arrays, and thus systematic SEC-DAEC codes. In order to construct the codes in accordance with the simple rules of Section I-B, however, it is necessary to modify slightly the codes we have just obtained.

This modification is accomplished in two steps.

1) Select A_1 as the $k - 1$-digit binary number, consisting of all ones, except for a zero in the first digit. This step will insure that we select a symmetric check array ending in all ones.
2) After obtaining the symmetric check array ending in all ones, add C to every A_i of this check array, where C consists of $k - 1$ ones. The array obtained from this operation will still be a check array [that is, it satisfies a')-c')], but it will not be a symmetric check array. For this check array, the binary number not included in the A_i column is composed of all *ones*, while the binary number not included in the B_i column is composed of all *zeros*. We shall call such a check array an *antisymmetric check array.*

Now the key property possessed by the particular antisymmetric check array, which we construct with steps 1 and 2, is the composition of A_n, A_{n-1}, A_{n-2}, $\cdots$ $A_{n-(k-2)}$. The $k - 1$ digits of A_n are all zeros; the first $k - 2$ digits of A_{n-1} are zeros; the first $k - 3$ digits of A_{n-2} are zeros; and so on, until we reach $A_{n-(k-2)}$, whose first digit is a zero. We recall that A_i consists of the first $k - 1$ digits of the checking number which indicates that the ith digit of the word is in error. When we construct the parity check table (see Fig. 3) corresponding to these A_i, then we find that: y_1 does not check any of the other y_i; y_2 does not check y_3, y_4, $\cdots$ y_{k-1}, or y_k; y_3 does not check y_4, y_5, $\cdots$ y_{k-1}, or y_k; and so on, until y_{k-1}, which does not check y_k.

A given parity check digit of these codes then may in general check any information digit and any *preceding* parity digit.

H. The Number of Different Codes

From Sections III-E and III-G, we see that any maximal-length linear binary shift register corresponds to a SEC-DAEC code. Since there does not appear to be any reason for preferring one such register over another with the same number of stages,[6] we have listed in Table III one register each, for $R = 3, 4, \cdots 19$, where R is the number of stages. Zierler [5] has shown that the total number of different R-stage maximal-length linear binary shift registers is given by

$$\frac{\phi(2^R - 1)}{R} \qquad (14)$$

where $\phi(k)$ is known as Euler's Phi-function. For any positive integer k, $\phi(k)$ is just the number of positive integers less than k and prime to k, 1 included. In Table V, we have given

$$\frac{\phi(2^R - 1)}{R}$$

for $R = 3, 4, \cdots 19$. (This table was obtained from Marsh's tables [8].)

[6] This neglects the fact that some registers require more mod-2 adders than others. Table III gives registers which require the minimum number of adders for $R < 13$.

TABLE V
THE NUMBER OF DIFFERENT R-STAGE MAXIMAL-LENGTH LINEAR BINARY SHIFT REGISTERS

R	$\frac{\phi(2^R - 1)}{R}$
3	2
4	2
5	6
6	6
7	18
8	16
9	48
10	60
11	176
12	144
13	630
14	756
15	1800
16	2048
17	7710
18	7776
19	27594

$[\phi(2^R - 1)/R]$ = number of different R-stage maximal-length linear binary shift registers.

I. Another Type of SEC-DAEC Code

The linear binary feedback shift register was introduced as a particular method of obtaining an antisymmetric check array (see Section III-G). It is, however, not the only method. An antisymmetric check array which cannot be obtained from a linear shift register[7] is given here.

$$\begin{array}{cc} 1\,0\,0\,0 & 0\,1\,0\,0 \\ 1\,1\,0\,0 & 0\,0\,1\,0 \\ 1\,1\,1\,0 & 1\,0\,0\,1 \\ 0\,1\,1\,1 & 1\,1\,0\,0 \\ 1\,0\,1\,1 & 1\,1\,1\,0 \\ 0\,1\,0\,1 & 1\,1\,1\,1 \\ 1\,0\,1\,0 & 0\,1\,1\,1 \\ 1\,1\,0\,1 & 1\,0\,1\,1 \\ 0\,1\,1\,0 & 0\,1\,0\,1 \\ 0\,0\,1\,1 & 1\,0\,1\,0 \\ 1\,0\,0\,1 & 1\,1\,0\,1 \\ 0\,1\,0\,0 & 0\,1\,1\,0 \\ 0\,0\,1\,0 & 0\,0\,1\,1 \\ 0\,0\,0\,1 & 0\,0\,0\,1 \\ 0\,0\,0\,0 & 1\,0\,0\,0 \end{array} \qquad (15)$$

This array defines a systematic SEC-DAEC code. The instrumentation of this code, however, appears to involve a good deal more equipment than the instrumentation of codes derived from linear shift registers.

J. Related Codes

It is instructive to view the class of codes obtained in this paper as a particular type of single-error-correcting-double-error-detecting (SEC-DED) code. Consider an SEC-DED code with m information digits, k parity digits, and $n = m + k$ total digits. We require not only that the code detect all double errors, but also that the possible checking numbers obtained at the output for this code shall divide the set of $(n)\ (n-1)/2$ possible double errors into n disjoint sets. Each of these disjoint sets shall contain one, and only one, double adjacent error (counting an error in the first and last digits as an adjacent error). It is easy to show that, for any k, the upper bound on m for such a code is just m^*, where m^* was defined in Section III-A. As a matter of fact, the SEC-DAEC codes obtained here are an example of this type of SEC-DED code.

The above paragraph suggests the possibility of obtaining single error correcting (SEC) codes derived from the SEC-DAEC codes given here. The construction of such SEC codes is almost trivial. We merely require that the checking numbers for such an SEC code be the complements[8] of the numbers in the flip-flops of a maximal-length linear binary shift register. The SEC codes obtained in this manner will require the same number of parity digits as ordinary Hamming codes.

APPENDIX

A PROPERTY OF m SEQUENCES

We wish to show (see Section III-F) that if

$$\begin{array}{c} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{array}$$

are the successive entries in the flip-flops of a maximal-length linear binary shift register, and

$$B_i = A_i + A_{i+1} \qquad i = 1, 2, \cdots n-1 \qquad (16)$$

$$B_n = A_n + A_1 \qquad (17)$$

then

$$B_i = A_{i+s} \qquad (18)$$

for

$$i = 1, 2, \cdots n$$

and some fixed s
$(i + s)$ is taken mod n.

To simplify the notation, we relabel as follows:

$$A_1 = A = (a_1\ a_2\ \cdots\ a_{k-1})$$

$$A_2 = B = (b_1\ b_2\ \cdots\ b_{k-1})$$

$$A_3 = C = (c_1\ c_2\ \cdots\ c_{k-1})$$

$$B_1 = D = (d_1\ d_2\ \cdots\ d_{k-1})$$

$$B_2 = E = (e_1\ e_2\ \cdots\ e_{k-1}).$$

[7] This check array was obtained by W. Michaels.

[8] The complement of a binary number is obtained by replacing all zeros in the number by ones and all ones by zeros.

Let the values of S_i (see Fig. 2) for the register producing $A_1, A_2, \cdots A_n$ be $S_1, S_2, \cdots S_{k-1}$. Then we must have

$$b_i = a_{i-1} \qquad i = 2, 3, \cdots k-1 \tag{19}$$

$$b_1 = \sum_{i=1}^{k-1} S_i a_i \tag{20}$$

and

$$C_i = b_{i-1} \qquad i = 2, 3, \cdots k-1 \tag{21}$$

$$C_1 = \sum_{i=1}^{k-1} S_i b_i. \tag{22}$$

Furthermore, we have from (16) and (17)

$$d_i = a_i + b_i \tag{23}$$

$$e_i = b_i + c_i. \tag{24}$$

Then

$$\begin{aligned} e_i &= b_1 + c_1 \\ &= \sum_{i=1}^{k-1} S_i a_i + \sum_{i=1}^{k-1} S_i b_i \\ &= \sum_{i=1}^{k-1} S_i (a_i + b_i) \\ &= \sum_{i=1}^{k-1} S_i d_i. \end{aligned} \tag{25}$$

Furthermore, for $i = 2, 3, \cdots k-1$

$$\begin{aligned} e_i &= b_i + c_i \\ &= a_{i-1} + b_{i-1} \\ &= d_{i-1}. \end{aligned} \tag{26}$$

Now, since the arguments given will apply to any two successive B_i, (25) and (26) show that the sequence of the B_i may be obtained from the same shift register as the A_i. Finally, since this is a maximal-length shift register, the sequence of the B_i must just be a shifted version of the sequence of the A_i, and (18) must hold.

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A Probabilistic Model for Run-Length Coding of Pictures*

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Summary—**A first-order Markoff process representation for pictures is proposed in order to study the picture coding system known as run-length coding (differential-coordinate encoding). A lower bound for the saving in channel capacity is calculated on the basis of this model, and is compared with the results obtained by previous investigators. In addition, this representation is shown to yield an insight into the run-length coding system which might not otherwise be obtained. The application of this probabilistic model to an "elastic" system of run-length coding is also discussed.**

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Introduction

IT has been recognized that a saving in either time or bandwidth is possible by exploiting the large-scale redundancies in pictures. One of the first proposals advanced for taking advantage of these redundancies is the variable sweep system of Cherry and Gouriet.[1] The authors found that theoretically a saving of approximately seven in either time or bandwidth is possible with their system.

[1] E. C. Cherry and G. G. Gouriet, "Some possibilities for the compression of television by recoding," *Proc. IEE*, vol. 100, pt. III, pp. 9–18; January, 1953.

Another scheme which has been proposed is the run-length coding, or differential-coordinate encoding system.[2] Although this system is applicable to half-tone pictures[3] we shall, for simplicity, consider its implementation only for black and white pictures. The basic idea in this scheme is to transmit only the lengths of the black and white runs in a picture as they occur in successive scanning lines. The first implementation of this system appears to be that of Treuhaft.[4] However, he gives no results concerning the possible reduction in either time or bandwidth for a run-length coding type system. Deutsch[5] has measured the probability distribution of runs in a section of typewritten material, and concluded that a saving of approximately two is possible in a run-length coding system which uses an optimum code. More extensive studies of the run-length probability distributions for typewritten material has been made by Michel[6] who concluded from his findings that theoretically a saving of approximately ten is possible with a run-length coding system which employs an optimum code.

These results[5,6] are sufficiently encouraging to warrant further investigation of the run-length coding system. The present work stems from a desire to predict more generally the saving that is possible with a run-length coding system for various types of black and white pictures, and to gain an insight into this system. The analysis is carried out only for black and white pictures, and for a binary digital transmission channel.

The Description of a Picture in Terms of a First-Order Markoff Process

The process of scanning reduces a picture from a two-dimensional array of cells (resolution elements) to a one-dimensional sequence of cells. In the case of a black and white picture such a sequence would consist of a succession of black and white cells. A section of this sequence might appear as below.

$$\cdots \text{BBBWWBBBBBBWWWWBWBBWBWW} \cdots$$

Thus, in our subsequent discussion, when we use the word "picture" we shall, in reality, be referring to a one-dimensional sequence of cells which results from scanning a picture. (*Note*: Since there are many ways to scan a picture, this representation is not unique. However, for our purposes, this is unimportant.)

The assumption, on which the analysis presented herein is based, can now be stated.

The transition in intensity from a given cell to the immediately succeeding cell is determined solely by the intensity of the given cell.

This dependence is a probabilitistic one which is given in terms of the transition probabilities $p_w(w)$, $p_w(b)$, $p_b(b)$, and $p_b(w)$, where[7]

$p_w(w)$ is the probability that a cell is white, given that the immediately preceding cell is white;

$p_w(b)$ is the probability that a cell is black, given that the immediately preceding cell is white

$$= 1 - p_w(w);$$

$p_b(b)$ is the probability that a cell is black, given that the immediately preceding cell is black;

$p_b(w)$ is the probability that a cell is white, given that the immediately preceding cell is black

$$= 1 - p_b(b).$$

Two other parameters which enter into the analysis are are $p(w)$, and $p(b)$, where[8]

$p(w)$ is the probability of obtaining a white cell;

$p(b)$ is the probability of obtaining a black cell

$$= 1 - p(w).$$

The transition probabilities can be related to each other, by noting first that

$$p(w; b) = p(b; w)$$

where

$p(w; b)$ = the joint probability of obtaining a white cell followed by a black cell;

$p(b; w)$ = the joint probability of obtaining a black cell followed by a white cell.

Hence, since $p(w; b) = p(w)p_w(b)$, and $p(b; w) = p(b)p_b(w)$, we obtain

$$p(w)p_w(b) = p(b)p_b(w).$$

It is easily seen that only two of the six probabilities $p(w)$, $p(b)$, $p_w(w)$, $p_w(b)$, $p_b(b)$, $p_b(w)$ are independent; for example, an independent set comprises $p(w)$ [or $p(b)$], and any one of the four transition probabilities.

At this point we note that the assumption is equivalent to stating that a picture can be represented by a first-order Markoff process with stationary transition probabilities.[9] This model for an information source has been proposed previously by Oliver,[10] among others.

[2] A. E. Laemmel, "Coding Processes for Bandwidth Reduction in Picture Transmission," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Rept. No. R 246–51; August, 1951.

[3] W. F. Schreiber and C. F. Knapp, "TV bandwidth reduction by digital coding," 1958 IRE National Convention Record, pt. 4, pp. 88–99.

[4] M. A. Treuhaft, "Description of a System for Transmission of Line Drawings with Bandwidth-Time Compression," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Rept. No. R 339-53; September 4, 1953.

[5] S. Deutsch, "Some Statistics Concerning Typewritten or Printed Material," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Rept. No. R-526; October 31, 1956.

[6] W. S. Michel, "Statistical encoding for text and picture communication," *Commun. and Electronics*, vol. 35, pp. 33–36; March, 1958.

[7] It is well to emphasize that the transition probabilities are not unique, since they depend on the particular scanning direction which is used. However, for a given scanning direction the transition probabilities are unique.

[8] $p(w)$ and $p(b)$ are unique for a particular picture, since they do not depend on the scanning direction.

[9] W. Feller, "An Introduction to Probability Theory and Its Applications," John Wiley and Sons, Inc., New York, N. Y., vol. 1, 2nd ed.; 1957.

[10] B. M. Oliver, "Efficient coding," *Bell Sys. Tech. J.*, vol. 31, pp. 724–750; July, 1952.

Since a transition from a given intensity to any other intensity is possible in a sufficiently large number of steps, it follows that our Markoff process representation for pictures is transitive, and thus ergodic. This fact will be of importance to us in our subsequent work.

It should be mentioned that our model does not take into account all of the dependence which exists among the various cells in the picture. Hence we can only expect that our results concerning information rates and savings in channel capacity are first-order approximations to the true values.

In particular, our results for the saving in channel capacity will at best be crude lower bounds. However, it should be borne in mind that a model which takes into account more of the dependences among the cells in the picture than ours will not be as amenable to analysis as our model.

Elias[11] has calculated the possible saving in channel capacity of a run-length coding system on the basis of a model in which successive cells in the picture are considered to be independent.

Number of Bits Required to Code a Picture by Means of a Nonstatistical Coding System

The present-day picture transmission method which specifies the intensity of each cell in a picture independently of the intensity of any other cell will be termed a nonstatistical coding system. This system does not exploit the redundancies which exist in a picture; as a consequence, the system is inefficient in the sense that it requires a larger number of bits to specify the picture than is actually necessary.

We now proceed to calculate the number of bits N_1 required by a nonstatistical coding system to specify a picture. Since the nonstatistical coding system specifies the intensity of each cell in a picture independently of all the other cells, it is clear that one bit is required for each cell in the picture. If we let N represent the total number of cells in the picture, then it is apparent that N_1 is equal to N.

It is advantageous to pause at this point to investigate the number N. If we consider that an average picture is $3\frac{1}{2}$ by 11 inches and that the resolution is 100 lines per inch in the vertical and horizontal directions, then N will be $(8.5)(11)(100)^2$, or approximately one million. Hence, we observe that N is very large, and indeed for our purposes may be considered to be infinite. We shall make use of this fact in our subsequent work by investigating the behavior of coding systems as N approaches infinity. Since N is very large, such a limiting behavior will tend to specify the true situation quite closely. In addition, taking limits as N approaches infinity enables us to simplify considerably the expressions which will be encountered.

[11] P. Elias, "Predictive coding," IRE Trans. on Information Theory, vol. IT-1, pp. 16–33; March, 1955.

Minimum Number of Bits Required to Code the Markoff Source Representation of a Picture

We now consider the minimum possible number of bits, $N_{\min}$, required by a statistical coding system to code the Markoff source representation of a picture. It is well known[12,13] that if the Markoff process is ergodic, and is described by the transition probabilities introduced previously, then the statistical coding system which achieves this minimum is one which codes the intensity of each cell on the basis of a knowledge of the intensity of the immediately preceding cell. If a binary alphabet is used, then it can be shown that $N_{\min}/N$ is equal to the conditional information content (conditional entropy) of the Markoff process; *i.e.*,[14]

$$\frac{N_{\min}}{N} = -p(b)p_b(b)\log p_b(b) - p(b)p_b(w)\log p_b(w) - p(w)p_w(w)\log p_w(w) - p(w)p_w(b)\log p_w(b). \quad (1)$$

We emphasize again that any coding system, whether statistical or nonstatistical, cannot code a picture with a fewer number of bits than $N_{\min}$. However, it is possible for a coding system to achieve the lower bound $N_{\min}$. In fact, as we shall see subsequently, the run-length coding system does achieve this lower bound as N approaches infinity.

Number of Bits Required by the Run-Length Coding System to Code the Markoff Process Representation of Pictures

Unlike the statistical coding scheme described above, which codes individual cells, the run-length coding system codes groups of cells.[15] Each group consists of an all-white or all-black linear array of cells. Such arrays or "runs," are readily found along the customary scanning lines. The run-length coding system counts the number of cells in each run and suitably encodes this number for transmission. If a binary alphabet is used in the coding process, then it can be shown[12] that the minimum number of bits required to code the run-lengths is given by the information content of the probability distribution of the run-lengths. Thus, in order to calculate the minimum number of bits required by the run-length coding system, we shall digress and find the probability distribution of the run-lengths. It should be pointed out that no specific codes are proposed (*e.g.*, Shannon-Fano-Huffman codes), but that only the minimum number of bits required to code an information source is computed.

[12] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, pp. 379–473, July, 1948; pp. 623–656, October, 1948.

[13] A. I. Khinchin, "Mathematical Foundations of Information Theory," Dover Publications, Inc., New York, N. Y., pp. 13–28; 1957.

[14] All logarithms are to the base two.

[15] W. S. Michel, W. O. Fleckenstein, and E. R. Kretzmer, "A coded facsimile system," 1957 WESCON Convention Record, pt. 2, pp. 84–93.

A white run of length x ($= 1, 2, \cdots$) is defined as a set of x consecutive white cells preceded by a black cell and succeeded by a black cell. A black run of length x is defined analogously.

Let R_w and R_b be the random variables which represent, respectively, the length of a white and a black run. In view of the assumption we have made, we see that as N approaches infinity, the probability of obtaining a white run of length equal to x is the same as the probability that $(x - 1)$ transitions from white to white have taken place, and that this is followed by a transition from white to black. Thus[16]

$$\text{prob}\,(R_w = x) = p_w(w)^{x-1}p_w(b), \qquad x = 1, 2, \cdots \quad (N \to \infty). \tag{2}$$

Similarly, it is obtained that the probability of obtaining a black run of length x is

$$\text{prob}\,(R_b = x) = p_b(b)^{x-1}p_b(w), \qquad x = 1, 2, \cdots \quad (N \to \infty). \tag{3}$$

If we let H_w denote the information content of the white run-length probability distribution, then

$$\begin{aligned} H_w &= -\sum_{x=1}^{\infty} \text{prob}\,(R_w = x) \log \text{prob}\,(R_w = x) \\ &= -\log\,[(1 - p_w(w))/p_w(w)] \\ &\qquad - [1 - p_w(w)]^{-1} \log p_w(w) \end{aligned} \tag{4}$$

where (4) is obtained by making use of the summation formulas

$$\sum_{x=0}^{\infty} a^x = (1 - a)^{-1} \qquad |\,a\,| < 1 \tag{5}$$

$$\sum_{x=0}^{\infty} xa^x = a(1 - a)^{-2} \qquad |\,a\,| < 1. \tag{6}$$

Similarly, if we let H_b denote the information content of the black run-length probability distribution, we obtain

$$\begin{aligned} H_b &= -\sum_{x=1}^{\infty} \text{prob}\,(R_b = x) \log \text{prob}\,(R_b = x) \\ &= -\log\,[[1 - p_b(b)]/p_b(b)] \\ &\qquad - [1 - p_b(b)]^{-1} \log p_b(b). \end{aligned} \tag{7}$$

At this point we note that one of the consequences of the assumption that N approaches infinity is that the run-lengths are independent. If N is finite then any set of i run-lengths must satisfy the inequality that the sum of their lengths be less than or equal to N; hence these run-lengths are dependent. If we allow N to approach infinity then any subset of k ($k < i$) run-lengths will satisfy the inequality independently of the other $k - i$ run-lengths; *i.e.*, the set of k run-lengths is independent of the set of $k - i$ run-lengths. Since i and k are arbitrary it follows that all of the run-lengths will be independent as N approaches infinity. This fact will be used in our subsequent discussion.

Since all the black and white runs which constitute the picture have the same probability distribution, it is obvious that the total number of bits required by the run-length coding system is obtained by summing the information contents H_w and H_b over the entire picture. We denote the total number of runs by n; since n is very large we can assume, without loss of generality, that n is even so that there are $n/2$ white and $n/2$ black runs in the picture. Hence the total number of bits N_R required to specify the Markoff process representation of the picture by run-length coding is

$$\begin{aligned} N_R &= (n/2)(H_w + H_b) \\ &= (n/2)(-\log\,((1 - p_w(w))/p_w(w)) \\ &\qquad - (1 - p_w(w))^{-1} \log p_w(w) \\ &\qquad - \log\,((1 - p_b(b))/p_b(b)) \\ &\qquad - (1 - p_b(b))^{-1} \log p_b(b)). \end{aligned} \tag{8}$$

We are now in a position to prove the following theorem.

Theorem: In the limit, as N approaches infinity, the number of bits N_R required to code the black and white runs in a first-order Markoff chain is equal to the theoretical lower bound $N_{\min}$.

Proof: From (1) and (8) we obtain

$$\begin{aligned} &N_R/N_{\min} \\ &\quad = \lim_{N \to \infty} (n/2N)((1 - p_w(w))^{-1} + (1 - p_b(b))^{-1}). \end{aligned} \tag{9}$$

We now digress to evaluate the $\lim_{N\to\infty} (n/2N)$. In order to do this we note that the sum of the lengths of the white and black run-lengths is equal to the number of cells in the picture. That is,

$$\sum_{i=1}^{n/2} (R_w^i + R_b^i) = N \tag{10}$$

where R_w^i and R_b^i are defined as the random variables which represent, respectively, the length of the ith white and ith black runs.

Eq. (10) can be rewritten as follows:

$$(2/n) \sum_{i=1}^{n/2} (R_w^i + R_b^i) = 2N/n. \tag{11}$$

However, $(2/n) \sum_{i=1}^{n/2} R_w^i$ is a sum of independent and identically distributed random variables; hence, by the Kolmogoroff Strong Law of Large Numbers[17] (as $N \to \infty$), this sum converges with probability one to the expected value of the white run lengths, $E(R_w)$ [if $E(R_w) < \infty$]. Similarly, $(2/n) \sum_{i=1}^{n/2} R_b^i$ converges to $E(R_b)$ [if $E(R_b) < \infty$], with probability one, as N approaches infinity. Thus,

$$\lim_{N \to \infty} (2N/n) = E(R_w) + E(R_b) \tag{12}$$

[16] The probability distribution of R_w, as well as that of R_b, is known as a geometric distribution.[9]

[17] M. Loeve, "Probability Theory," D. Van Nostrand Co., Inc., Princeton, N. J.; 1955.

hich, by (2) and (3) and the summation formula in 5), becomes

$$= [[1 - p_w(w)]^{-1} + [1 - p_b(b)]^{-1}]. \quad (13)$$

ubstituting (13) into (9), we obtain

$$N_R/N_{\min} = 1$$

hich proves the theorem.

The result stated by our theorem is not surprising, ince both N_R and $N_{\min}$ represent the number of bits equired to specify N cells in the first-order Markoff hain. The quantity $N_{\min}$ is calculated on the basis of ndividual cells and N_R is calculated on the basis of groups f cells which become independent as N approaches nfinity. Thus N_R and $N_{\min}$ must be the same as N pproaches infinity. The fact that the groups of cells are ndependent is important; this will be seen to be true hen we consider a run-length coding system which ses the same code for black and white runs of the same ength.

Saving in Channel Capacity

We now turn our attention to the calculation of the aving in channel capacity S of the run-length coding ystem with respect to the non-statistical coding scheme. t is apparent that S is given by

$$S = N/N_R. \quad (14)$$

n view of (1) and the theorem which we have just roved, (14) becomes

$$S = N/N_R = N/N_{\min}$$
$$= (-p(b)p_b(b) \log p_b(b) - p(b)p_b(w) \log p_b(w)$$
$$- p(w)p_w(w) \log p_w(w) - p(w)p_w(b) \log p_w(b))^{-1}. \quad (15)$$

Since $N_{\min}$ is less than or equal to N for all values of (w) and of transition probabilities, it follows that S s always greater than or equal to unity, for all types of ictures. Thus, our probabilistic model predicts that he saving in channel capacity is always greater than r equal to unity for all possible pictures. For a given ransmission facility the saving in channel capacity ecomes a saving in time. For a given transmission time, he saving in channel capacity becomes a saving in andwidth.

A graph of the saving is shown in Figs. 1(a) and 1(b), here for the sake of convenience the independent ariables are chosen to be $p(w)$ and $p_w(w)$. It is easily ound that for a fixed $p(w)$, the minimum value of S, enoted by $S_{\min}(p(w))$, occurs when $p_w(w) = p(w)$, and s given by

$$S_{\min}(p(w)) = (-p(w) \log p(w) - p(b) \log p(b))^{-1}. \quad (16)$$

The minimum possible value of $S_{\min}(p(w))$ occurs when $(w) = \frac{1}{2}$, and is equal to one. Thus, the saving is one for ictures which contain equal black and white areas, and ave a transition probability $p_w(w)$ of $\frac{1}{2}$. In such cases,

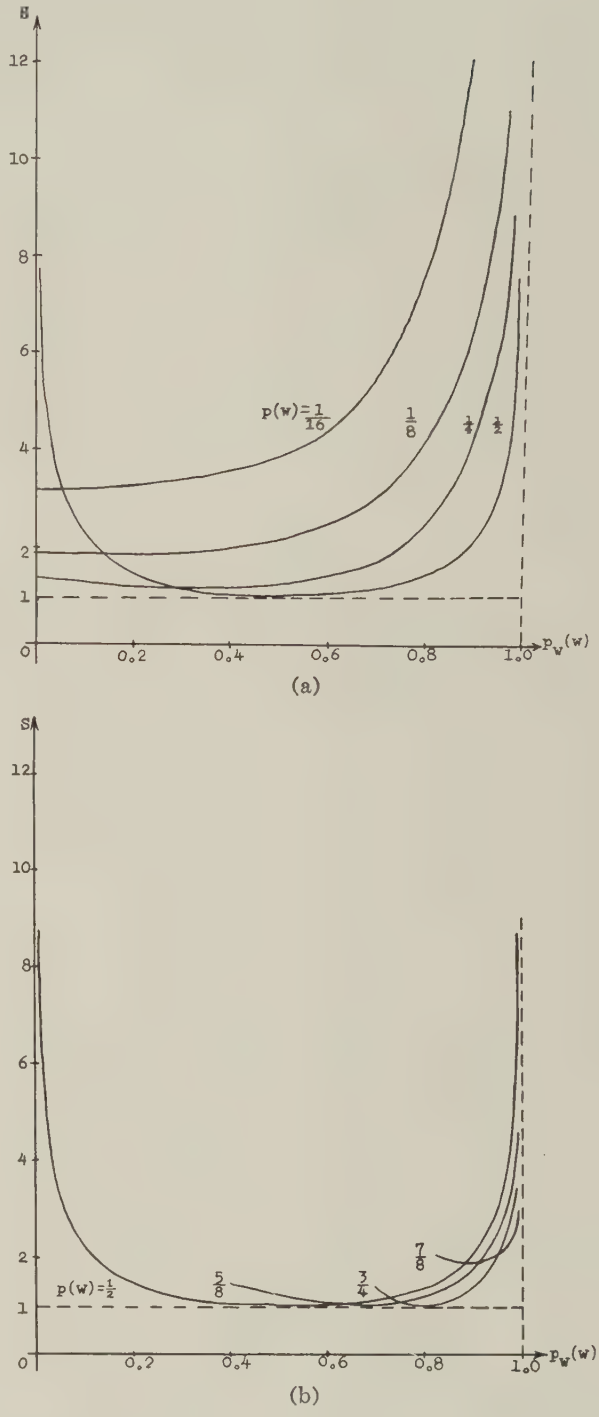


Fig. 1—(a) Saving in channel capacity vs transition probability $p_w(w)$, for various values of $p(w)$ between 0 and $\frac{1}{2}$. (b) Saving in channel capacity vs transition probability $p_w(w)$, for various values of $p(w)$ between $\frac{1}{2}$ and 1.

there would be no point in using a statistical coding system, and we would no doubt use a nonstatistical coding system. However, such pictures very rarely occur in practice. In fact, such a picture would be the equivalent of a random pattern, in which there is no dependence among the various cells. Hence our probabilistic model predicts that for pictures which are equivalent to random patterns there is no point in using a run-length coding system, since there is very little to be gained.

We turn our attention now to the question of when there are large savings. From Fig. 1(a) and 1(b) we observe that large savings are possible for the following three cases:

1) If $p_w(w)$ is close to unity, then the saving will be large, regardless of the value of $p(w)$. This corresponds to the situation where the white runs in the picture are very long. In this case a very large saving can be expected.

2) If $p(w)$ is close to zero, or one, then the saving is again very large, regardless of the value of $p_w(w)$. This situation corresponds to a picture which is almost completely black, or completely white, and again it is to be expected that a large saving is to be obtained by coding runs rather than using a nonstatistical transmission method.

3) If $p_w(w)$ is close to zero, and $p(w)$ is close to one-half, then the saving is again very large. The situation is that corresponding to a picture which is a checkerboard pattern. Thus, since it is known that all black and white runs in the picture will be of unit length, the transmitter need send no bits to specify the picture (except possibly for one bit to specify whether the first cell is black or white).

We observe that for given values of $p(w)$ the curves of Fig. 1(b) do not extend beyond certain critical values of $p_w(w)$. This is so, since values of $p_w(w)$ less than these critical values lead to impossible values for the other transition possibilities.

Comparison with Deutsch's Results

It is of interest to compare the saving in channel capacity predicted by the probabilistic model with the value of saving found by previous investigators who have obtained that saving by actually measuring the probability distribution of the run-lengths in a picture. Towards this end, we compare our results with those of Deutsch,[5] who has measured the run-length probability distribution for the two cases of horizontally and vertically scanning a picture. From his results we obtain that

$$p(w) = 4107/5025,$$

and that the transition probability $p_w(w)$ in the vertical direction is

$$p_w(w) = 3702/4107 \qquad \text{(vertical direction)}.$$

Substituting these values in (15) we obtain

$$S = 1.78 \qquad \text{(vertical direction)}.$$

The corresponding value obtained by Deutsch is 2.02.

The transition probability in the horizontal direction is

$$p_w(w) = 3640/4107 \qquad \text{(horizontal direction)}.$$

The value of $p(w)$ is of course the same for both the vertical and horizontal directions. Substituting these values in (15), we obtain

$$S = 1.67 \qquad \text{(horizontal direction)}.$$

The corresponding value found by Deutsch is 2.14. Thus the saving in channel capacity predicted by our model is in error by approximately 12 per cent in the vertical direction and about 22 per cent in the horizontal direction. In both cases the probabilistic model predicts a saving which is less than the true value. In view of the rather close agreement found for the value of the saving we would feel that the probabilistic model is a good one for predicting the amount of saving possible for the run-length coding system. However, more results of this kind would certainly have to be obtained before any general statement to this effect could be made.

The Use of the Same Code for Black and White Runs of the Same Length vs Different Code for Black and White Runs of the Same Length

It is certainly simpler to implement a run-length coding system which uses the same code for black and white runs of the same length than one which uses a different code for black and white runs of the same length. However, we have seen that our model predicts that this latter method achieves the lower bound of (1), as N approaches infinity, and thus offers a greater saving than the former method. We now investigate how much more of a saving is possible with this latter method than with the former. In order to do this we must compute the information content of the probability distribution of a run-length which can be either white or black.

Let R denote the random variable which represents the length of either black or white runs. Then, by reasoning which is similar to that used in obtaining (2) we find

$$\text{prob}\,(R = x) = p(w)p_w(w)^{x-1}p_w(b) + p(b)p_b(b)^{x-1}p_b(w) \qquad x = 1, 2, \cdots \quad (N \to \infty). \tag{17}$$

Denoting the information content of this probability distribution by H_S, we find that

$$H_S = \sum_{x=1}^{\infty} \left(-p(w)p_w(w)^{x-1}p_w(b) - p(b)p_b(b)^{x-1}p_b(w)\right) \log \cdot \left(p(w)p_w(w)^{x-1}p_w(b) + p(b)p_b(b)^{x-1}p_b(w)\right). \tag{18}$$

If the number of bits required to code the picture by this method is denoted by N_S, then

$$N_S = nH_S. \tag{19}$$

It is easily seen that if $p(w) = p(b) = \frac{1}{2}$, then $N_S = N_{\min}$. Thus this method of coding achieves the lower bound of (1) for pictures which contain equal black and white areas. Hence for pictures which contain roughly the same number of black and white cells there is no advantage to be gained in coding black and white runs of the same length with different codes, instead of coding them with the same code.

If $p(w)$ [or $p(b)$] is close to zero, then it is easily seen that H_S is approximately twice as large as $H_{\min}$. Thus in this case there is an advantage to be gained in using a

fferent code for black and white runs of the same ngth, rather than using the same code.

A straightforward calculation shows that run-lengths this coding scheme are no longer independent, except hen $p(w) = p(b) = \frac{1}{2}$. This accounts for the fact that s and $N_{\min}$ are different for those pictures which do not ave the same number of black and white cells.

Exponential Character of the Run-Length Probability Distribution

The probability distribution of the black and white uns [(2) and (3)] as predicted by our probabilistic model r pictures has an exponential character, as shown in ig. 2. This result also bears some resemblance to the sults for the probability distributions measured by her investigators. Michel[6] finds a probability distri- ution of runs which has a very strong exponential trend. me of the results obtained by Deutsch[5] also indicate strong exponential trend. However, some of his results o not indicate such a trend. In particular, the probability istribution that he finds for the white runs in the vertical irection does not indicate an exponential trend, and in ct exhibits no trends.

Application of the Probabilistic Model to an "Elastic" System of Run-Length Coding

It will be recalled that the saving in channel capacity calculated for a particular probability distribution of un-lengths. Thus, the saving pertains only to that set of ictures which has the aforementioned probability dis- ribution. If a picture with a different probability dis- ribution is coded by the run-length coding system, then is possible that there will be no saving in channel apacity. That is, if the code is not "matched" to the robability distribution of run-lengths in a particular icture, there may not be a saving in channel capacity.[6] his is one of the disadvantages of the run-length coding ystem. One method of solving this problem would be to neasure the run-length probability distribution of the icture before the picture is coded, in order to match the ode to this distribution. Such a system, which changes s code for different pictures, is known as an "elastic" ystem. In practice, an elastic system which measures the un-length probability distributions directly is difficult to implement, since the measurement of run-length probability distributions is difficult to make.

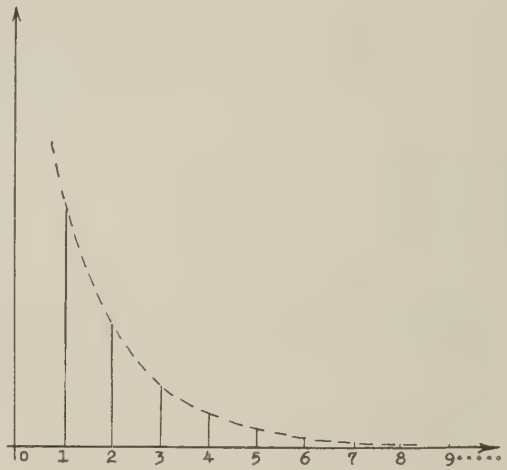


Fig. 2—Exponential character of the probability distribution of run-lengths.

As an alternative to the above method there is the possibility of computing the probability distribution of run-lengths on the basis of the probabilistic model described previously. The outstanding feature of this model is that it is described by parameters which are very simple to measure in practice. The transition probabilities which describe the model can be measured very easily by means of the optical correlator used by Kretzmer[18] and other investigators[19] in their studies concerning the measurements of the statistics of pictures. Once these transition probabilities have been measured, the elastic run-length coding system could match its code to the run-length probability distributions which are calculated from it. In this manner, the elastic run-length coding system could achieve a greater saving in channel capacity, on the average, than would be otherwise possible.

Acknowledgment

The author would like to thank Professor Lotfi A. Zadeh and Dr. Ernest R. Kretzmer for many helpful discussions and criticisms concerning this material.

[18] E. R. Kretzmer, "Statistics of television signals," *Bell Sys. Tech. J.*, vol. 31, pp. 751–763; July, 1952.

[19] J. Capon, "Bounds to the Entropy of Television Signals," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rept. No. 296; May 25, 1955.

A Note on Invariant Relations for Ambiguity and Distance Functions*

C. A. STUTT†

Summary—Woodward's result for the ambiguity function, that the volume associated with its squared magnitude over the time-shift and frequency-shift plane is a constant, has been shown to be true also for a cross-ambiguity function for two time functions. If complex time functions have been obtained by means of a Hilbert transformation from real time functions, it is found for the cross-ambiguity function that the volumes under the squared real part and under the squared imaginary part are constant and contribute equally to the volume under the squared magnitude function. A "distance" function for two time functions is defined to be the integrated squared difference between these functions. The relation for the squared real part of the ambiguity function readily yields an invariant relation for the volume associated with this distance function in the case of Hilbert-derived complex time functions. An especially simple invariant relation for the "mean" distance, as computed over the time-shift and frequency-shift plane, exists for such time functions having finite energy and finite mean value.

I. Invariant Relation for the Ambiguity Function

ONE of the problems in radar is to identify unambiguously two or more targets which may be arbitrarily close to each other both in range and velocity. In order to describe those attributes of a radar signal which determine how well such an identification can be made, Woodward has proposed the use of the correlation function for a combined time shift and frequency shift, which had been introduced earlier by Ville.[1,2] This function, which is frequently referred to as the *ambiguity function*, is defined in terms of a time-shift variable τ and a frequency-shift variable Δ as

$$\chi(\tau, \Delta) = \int_{-\infty}^{\infty} u(t)u^*(t + \tau)e^{-2\pi i\Delta t}\, dt \tag{1}$$

$$= \int_{-\infty}^{\infty} U^*(f)U(f + \Delta)e^{-2\pi i f\tau}\, df \tag{2}$$

where $u(t)$ is the complex low-frequency modulating waveform of the radar signal and $U(f)$ its Fourier transform. Eq. (1) expresses $\chi(\tau, \Delta)$ as a Fourier transformation of the time function $u(t)\ u^*(t + \tau)$ into the Δ domain with τ a parameter, while (2) expresses it as a Fourier transformation of the frequency function $U^*(f)\ U(f + \Delta)$ into the τ domain with Δ a parameter.

A fundamental result given by Woodward for the ambiguity function is that the volume under the surface described by $|\chi(\tau, \Delta)|^2$ is a constant, *i.e.*,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\ \chi(\tau, \Delta)\ |^2\, d\tau\, d\Delta = 1 \tag{3}$$

where that constant is unity if $u(t)$ is normalized so that

$$\int_{-\infty}^{\infty} |\ u(t)\ |^2\, dt = 1. \tag{4}$$

Further studies of the ambiguity function, with particular reference to the radar waveform design problem, have been made by Siebert and Lerner.[3-5]

II. Invariant Relation for the Cross-Ambiguity Function

The preceding result may be said to apply to the *auto-ambiguity function*. If a *cross-ambiguity function* is defined for two complex waveforms $u_1(t)$ and $u_2(t)$ and suitable normalizations are made, a result identical to (3) may be readily shown to exist. Thus, the function $\chi_{12}\ (\tau, \Delta)$ is defined by

$$\chi_{12}(\tau, \Delta) = \int_{-\infty}^{\infty} u_1(t)u_2^*(t + \tau)e^{-2\pi i\Delta t}\, dt \tag{5}$$

which, when written in terms of the corresponding frequency functions $U_1(f)$ and $U_2(f)$ is

$$\chi_{12}(\tau, \Delta) = \int_{-\infty}^{\infty} U_2^*(f)U_1(f + \Delta)e^{-2\pi i f\tau}\, df. \tag{6}$$

Inasmuch as χ_{12} is the Fourier transform of $u_1(t)\ u_2^*(t + \tau)$, Parseval's theorem may be conveniently used to find the volume under the squared amplitude of χ_{12}; thus

$$\int_{-\infty}^{\infty} |\ \chi_{12}(\tau, \Delta)\ |^2\, d\Delta = \int_{-\infty}^{\infty} |\ u_1(t)u_2^*(t + \tau)\ |^2\, dt$$

so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\ \chi_{12}(\tau, \Delta)\ |^2\, d\tau\, d\Delta = \int_{-\infty}^{\infty} |\ u_1(t)\ |^2\, dt \int_{-\infty}^{\infty} |\ u_2(x)\ |^2\, dx \tag{7}$$

where $x = t + \tau$ in the last integral.

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[1] P. M. Woodward, "Probability and Information Theory, with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 7 and sec. 2.9; 1953.

[2] J. Ville, "Théorie et application de la notion signal analytique," *Cables and Transmission*, vol. 2, pp. 61–74; January, 1948.

[3] W. M. Siebert, "A radar detection philosophy," IRE Trans. on Information Theory, vol. IT-2, pp. 204–221; September, 1956.

[4] W. M. Siebert, "Studies of Woodward's Uncertainty Function," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Quart. Progr. Rept., pp. 90–94; April, 1958.

[5] R. M. Lerner, "Signals with uniform ambiguity functions," 1958 IRE National Convention Record, pt. 4, pp. 27–36.

With the energy in the waveforms normalized to unity,

$$\int_{-\infty}^{\infty} |\, u_1(t) \,|^2\, dt = \int_{-\infty}^{\infty} |\, u_2(t) \,|^2\, dt = 1 \tag{8}$$

e volume under $|\chi_{12}|^2$ becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\, \chi_{12}(\tau, \Delta) \,|^2\, d\tau\, d\Delta = 1. \tag{9}$$

The cross-ambiguity function (5), and the invariant lation (9), may be useful in the study of waveforms r communication systems wherein it is desired to identify a receiver one of many possible transmitted waveforms. such systems it is desirable to have waveforms which n be identified correctly, with high probability, despite ne shifts and frequency (Doppler) shifts which may ot be known at the receiver. Thus the distribution of this mbined correlation function of waveforms over the ne-frequency plane is of interest.

III. Invariant Relations for the Real and Imaginary Parts of $\chi_{12}(\tau, \Delta)$

It will now be shown that invariant relations similar to) also exist for the real part and for the imaginary art of χ_{12} in those cases for which the complex time nctions $u_1(t)$ and $u_2(t)$ are obtained from real time nctions by means of the Hilbert transform.[1] The real rt of χ_{12} is defined by

$$\zeta_{12}(\tau, \Delta) = \tfrac{1}{2}[\chi_{12}(\tau, \Delta) + \chi_{12}^*(\tau, \Delta)] \tag{10}$$

nd the imaginary part by

$$\xi(\tau, \Delta) = \frac{i}{2}\,[\chi_{12}^*(\tau, \Delta) - \chi_{12}(\tau, \Delta)]. \tag{11}$$

Then written in terms of ζ_{12} and ξ_{12}, (9) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\zeta_{12}^2 + \xi^2]\, d\tau\, d\Delta = 1. \tag{12}$$

rom (9) and (10) the volume under ζ_{12}^2 is

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta_{12}^2(\tau, \Delta)\, d\tau\, d\Delta \\
&= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\chi_{12}^2 + \chi_{12}^{*2} + 2\,|\,\chi_{12}\,|^2)\, d\tau\, d\Delta \\
&= \frac{1}{2} + \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{12}^2(\tau, \Delta)\, d\tau\, d\Delta \\
&+ \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{12}^{*2}(\tau, \Delta)\, d\tau\, d\Delta.
\end{aligned} \tag{13}$$

If the second integral on the right is written in terms f the frequency (spectrum) functions it becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{12}^{*2}(\tau, \Delta)\, d\tau\, d\Delta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau\, d\Delta \Big\{ \int_{-\infty}^{\infty} U_2(f) U_1^*(f + \Delta) e^{2\pi i f\tau}\, df \\
&\cdot \int_{-\infty}^{\infty} U_2(g) U_1^*(g + \Delta) e^{2\pi i g\tau}\, dg \Big\} \\
&= \int_{-\infty}^{\infty} d\Delta \Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_2(f) U_1^*(f + \Delta) U_2(g) \\
&\cdot U_1^*(g + \Delta)\, \delta(-f - g)\, df\, dg \Big\} \\
&= \int_{-\infty}^{\infty} U_2(f) U_2(-f)\, df \\
&\cdot \int_{-\infty}^{\infty} U_1^*(\Delta + f) U_1^*(\Delta - f)\, d\Delta
\end{aligned} \tag{14}$$

where $\delta(f)$ is a unit impulse function at $f = 0$.

For the case in which $u_2(t) = x_2(t) + iy_2(t)$ is a Hilbert-derived function, *i.e.*,

$$y_2(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_2(s)}{t - s}\, ds, \tag{15}$$

the value of the double integral in (14) is zero. This statement follows from the fact that in this case

$$U_2(f) = \begin{cases} 0 & f < 0 \\ 2X_2(f) & f \geq 0 \end{cases} \tag{16}$$

where $X_2(f)$ is the spectrum function associated with $x_2(t)$. Thus, with the exception of the point $f = 0$, the two factors $U_2(+ f)$ and $U_2(- f)$ in the integration with respect to f are nonoverlapping, *i.e.*, where one is nonzero the other is zero. The integration with respect to Δ produces a third function of f, which, being finite,[6] cannot produce a nonzero value in the f integral when the product $U_2(f)\ U_2(- f)$ is everywhere zero except possibly at the point $f = 0$. The case in which $x_2(t)$ has an average value, *i.e.*, an impulse occurs in $X_2(f)$ at $f = 0$ is excluded.

Identical arguments can be made for the first integral on the right in (13), so that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{12}^2(\tau, \Delta)\, d\tau\, d\Delta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{12}^{*2}(\tau, \Delta)\, d\tau\, d\Delta = 0.
\end{aligned} \tag{17}$$

[6] It is easily shown that $\int_{-\infty}^{\infty} U_1^*(\Delta + f) U_1^*(\Delta - f)\, d\Delta$ is finite since $U_1(f)$ belongs to L^2 if $u_1(t)$ does. If $U_1^*(\Delta + f)$ is denoted by $\psi_1(\Delta)$ and $U_1^*(\Delta - f)$ is denoted by $\psi_2(\Delta)$, then from Schwartz's inequality

$$\left|\, \int_{-\infty}^{\infty} \psi_1(\Delta)\psi_2(\Delta)\, d\Delta \,\right|^2 \leq \int_{-\infty}^{\infty} |\, \psi_1(\Delta) \,|^2\, d\Delta \int_{-\infty}^{\infty} |\, \psi_2(\Delta) \,|^2\, d\Delta.$$

But both integrals on the right exist since ψ_1 and ψ_2 belong to L^2, hence $|\int_{-\infty}^{\infty} \psi_1\psi_2\, d\Delta\,|^2$ and $\int_{-\infty}^{\infty} \psi_1\psi_2\, d\Delta$ are finite.

The volume under ζ_{12}^2 for Hilbert-derived complex time functions is now

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta_{12}^2(\tau, \Delta)\, d\tau\, d\Delta = \tfrac{1}{2}. \tag{18}$$

From (12) it follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^2(\tau, \Delta)\, d\tau\, d\Delta = \tfrac{1}{2}. \tag{19}$$

Eqs. (18) and (19) are the desired invariant relations for the real and imaginary parts of χ_{12}, and it is seen that they contribute equally to the volume under $|\chi_{12}|^2$ in the case of Hilbert-derived complex time functions.

IV. Invariant Relations for the "Distance" Between Time Functions

In the problem of waveform identification, which was mentioned in Section II, the concept of "distance" between two waveforms is also a useful one. For equal energy signals, the cross-correlation and distance functions are equivalent; however, the distance concept is somewhat easier to interpret geometrically. Thus if signals have a finite time duration T and a frequency bandwidth W,[7] they may be represented as points in an $N = 2TW$ dimensional space. Presumably, the separation of these points in the space would be optimized in some manner in the coding procedure for a common time origin. Timing may not be known at the receiver, however, and it is then necessary in the identification procedure to compare the received waveform with all the reference waveforms for all possible time shifts. These shifts may be interpreted geometrically as a relative motion of the signal points, and in this motion the distances can be changed radically. The picture may be further complicated by frequency shifts, additive noise, and random multipath effects. An objective in the coding procedure, then, is to choose the signal points so that they will not be moved close enough to each other to cause errors or ambiguities in their identification at the receiver.

In this section, the distance, or more precisely, the squared distance, between two time functions is defined to be the integrated squared difference between them. This distance is a function of the relative time shift τ and frequency shift Δ, and for finite energy signals is given by

$$\begin{aligned} d_{12}^2(\tau, \Delta) &= \int_{-\infty}^{\infty} |\, u_1(t; 0) - u_2(t + \tau; \Delta)\, |^2\, dt \\ &= \int_{-\infty}^{\infty} [u_1(t; 0) - u_2(t + \tau; \Delta)][u_1^*(t; 0) \\ &\qquad - u_2^*(t + \tau, \Delta)]\, dt \end{aligned} \tag{20}$$

where Δ is placed after the semicolon as a parameter of the time functions.

[7] These conditions are not incompatible if *finite time duration* means that the sample values of the time function are zero outside some time interval.

It will be useful in the manipulation of this equation to express $u_1(t)$ and $u_2(t)$ in terms of their amplitude and phase functions,

$$u_1(t) \equiv a_1(t)e^{i\phi_1(t)} \tag{21}$$

$$u_2(t) \equiv a_2(t)e^{i\phi_2(t)}, \tag{22}$$

so that a time-shifted and frequency-shifted signal becomes

$$u(t + \tau; \Delta) = a(t + \tau)e^{i[\phi(t+\tau)+2\pi\Delta(t+\tau)]}. \tag{23}$$

The four terms arising from the expansion of (20) are now

$$\begin{aligned} \int_{-\infty}^{\infty} u_1(t; 0)u_1^*(t; 0)\, d\tau &= \chi_{11}(0, 0) \\ &= \int_{-\infty}^{\infty} |\, u_1(t)\, |^2\, dt = 1, \end{aligned} \tag{24}$$

$$\begin{aligned} \int_{-\infty}^{\infty} u_2(t + \tau; \Delta)u_2^*(t + \tau; \Delta)\, dt \\ &= \int_{-\infty}^{\infty} a_2(t + \tau)e^{i[\phi_2(t+\tau)+2\pi\Delta(t+\tau)]} \\ &\quad \cdot a_2(t + \tau)e^{-i[\phi_2(t+\tau)+2\pi\Delta(t+\tau)]}\, dt \\ &= \int_{-\infty}^{\infty} a_2^2(t + \tau)\, dt \\ &= \chi_{22}(0, 0) = 1, \end{aligned} \tag{25}$$

$$\begin{aligned} \int_{-\infty}^{\infty} u_1(t; 0)u_2^*(t + \tau; \Delta)\, dt \\ &= \int_{-\infty}^{\infty} a_1(t)a_2(t + \tau)e^{i[\phi_1(t)-\phi_2(t+\tau)-2\pi\Delta(t+\tau)]}\, dt \\ &= \int_{-\infty}^{\infty} a_2(x)a_1(x - \tau)e^{-i[\phi_2(x)-\phi_1(x-\tau)]}e^{-2\pi i\Delta x}\, dx \\ &= \int_{-\infty}^{\infty} u_2^*(x)u_1(x - \tau)e^{-2\pi i\Delta x}\, dx \\ &= \chi_{21}^*(-\tau, -\Delta), \end{aligned} \tag{26}$$

$$\begin{aligned} \int_{-\infty}^{\infty} u_1^*\ (t; 0)u_2(t + \tau; \Delta)\, dt \qquad &\text{(conjugate of above)} \\ &= \chi_{21}(-\tau, -\Delta). \end{aligned} \tag{27}$$

In view of (24)–(27) and (10), (20) for $d_{12}^2(\tau, \Delta)$ becomes

$$\begin{aligned} d_{12}^2(\tau, \Delta) &= \chi_{11}(0) + \chi_{22}(0) \\ &\quad - \chi_{21}^*(-\tau, -\Delta) - \chi_{21}(-\tau, -\Delta) \\ &= 2 - 2\zeta_{21}(-\tau, -\Delta). \end{aligned} \tag{28}$$

The invariant relation for ζ_{12}^2 (18) applies equally well for ζ_{21}^2, hence an invariant relation involving d_{12}^2 is readily obtained:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \frac{d_{12}^2(\tau, \Delta)}{2}\right)^2 d\tau\, d\Delta = \tfrac{1}{2} \tag{29}$$

hich relation, again is valid for Hilbert-derived complex me functions.

One might inquire if the volume under d_{12}^2 is invariant ith respect to choice of time functions. In this con- ection it will be convenient to consider the mean value f d_{12}^2 over the plane, which is

$$\overline{d_{12}^2(\tau, \Delta)} = \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{4TW} \int_{-T}^{T} \int_{-W}^{W} d_{12}^2(\tau, \Delta)\, d\tau\, d\Delta$$

$$= 2 \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{4TW} \int_{-T}^{T} \int_{-W}^{W} [1 - \zeta_{21}(-\tau, -\Delta)]\, d\gamma\, d\Delta. \quad (30)$$

The mean value of $\zeta_{21}(-\tau, -\Delta)$, which appears on the ight-hand side of (30), is certainly the same as that for $_{21}(\tau, \Delta)$, and may be found in the following manner:

$$\overline{\zeta_{21}(\tau, \Delta)} = \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{4TW} \int_{-T}^{T} \int_{-W}^{W} \zeta_{21}(\tau, \Delta)\, d\tau\, d\Delta$$

$$= \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{8TW} \int_{-T}^{T} d\tau \int_{-W}^{W} d\Delta$$

$$\cdot \int_{-\infty}^{\infty} \{u_2(t)u_1^*(t + \tau)e^{-2\pi i\Delta t}$$

$$+ u_2^*(t)u_1(t + \tau)e^{2\pi i\Delta t}\}\, dt$$

$$= \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{8TW} \int_{-T}^{T} \int_{-\infty}^{\infty} \{u_2(t)u_1^*(t + \tau)$$

$$+ u_2^*(t)u_1(t + \tau)\} \frac{\sin 2\pi Wt}{\pi t}\, dt\, d\tau. \quad (31)$$

Now by the Fourier single-integral formula,[8]

$$f(x) = \lim_{W\to\infty} \int_{-\infty}^{\infty} f(t) \frac{\sin 2\pi W(x - t)}{\pi(x - t)}\, dt,$$

he integration with respect to t yields

$$\lim_{W\to\infty} \int_{-\infty}^{\infty} \{u_2(t)u_1^*(t + \tau) + u_2^*(t)u_1(t + \tau)\} \frac{\sin 2\pi Wt}{\pi t}\, dt$$

$$= \{u_2(0)u_1^*(\tau) + u_2^*(0)u_1(\tau)\} \quad (32)$$

[8] For example, E. C. Titchmarsh, "Introduction to the Theory f Fourier Integrals," Oxford University Press, New York, N. Y., nd ed., sec. 1.1; 1937.

where the term in braces corresponds to $f(t)$ and x is set equal to zero. The mean value of $\zeta_{12}(\tau, \Delta)$ in (31) is now

$$\overline{\zeta_{21}(\tau, \Delta)} = \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{8TW}$$

$$\cdot \int_{-T}^{T} \{u_2(0)u_1^*(\tau) + u_2^*(0)u_1(\tau)\}\, d\tau. \quad (33)$$

For time functions having finite mean value, which includes all cases of practical interest, it is apparent that the value of $\overline{\zeta_{21}(\tau, \Delta)}$ is zero in (33), *i.e.*,

$$\overline{\zeta_{21}(\tau, \Delta)} = \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{4TW} \int_{-T}^{T} \int_{-W}^{W} \zeta_{21}(\tau, \Delta)\, d\tau\, d\Delta = 0. \quad (34)$$

In view of this result, the mean value of d_{12}^2 over the plane is simply

$$\overline{d_{12}^2(\tau, \Delta)} = 2 \lim_{\substack{T\to\infty \\ W\to\infty}} \frac{1}{4TW} \int_{-T}^{T} \int_{-W}^{W} d\tau\, d\Delta \quad (35)$$

$$= 2.$$

Eq. (35) is the desired invariant relation for the distance function for complex time functions having finite energy and finite mean value.

In the case of wide band signals of long duration, the result contained in (34) is approximately satisfied over a finite rectangle $2T$ by $2W$, *i.e.*, the contribution of $\zeta_{21}(-\tau, -\Delta)$ to the integral in (30) is small compared to the contribution of the constant term of unity in the integrand. Thus, an approximate expression for the volume under the distance function is,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d_{12}^2(\tau, \Delta)\, d\tau\, d\Delta \approx 8TW. \quad (36)$$

This volume is seen to be determined entirely by the number of dimensions of the signals. Ideally, then, to make the best use of this volume, the coding procedure should place the ridges and peaks of the topography of d_{12}^2 in the regions of the expected trajectories of the shifted signal in the τ, Δ plane, and it should place the valleys outside these regions.

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On Upper Bounds for Error Detecting and Error Correcting Codes of Finite Length*

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Summary—**Upper bounds for error detecting and error correcting codes are obtained in this paper. One upper bound is found by exploiting the geometrical model of coding introduced by Hamming. The volume of an appropriate geometrical body is compared with the volume of the unit cube, in getting the first upper bound. An improvement on this upper bound can be found by introducing a mass density function, and comparing the mass of the body with the mass of the unit cube.**

A comparison is made with known upper bounds, and with best codes found thus far. The improved upper bound given here is frequently somewhat smaller than previously known upper bounds.

I. Introduction

A NUMBER of attempts[1-8] have been made to construct codes which will enable one to detect or correct errors on reception of a set of transmitted symbols. These attempts have been only partly successful; codes have been found which will be error detecting or correcting for particular lengths of the blocks of symbols used and for a highly restricted number of errors. No general procedures for obtaining codes have been discovered as yet, nor is it known, except in particular cases, whether a code is optimal or not.

Upper bounds for optimal binary codes of finite length have been found,[1] but these bounds are known to yield crude estimates of the number of possible code points in some cases, where the optimal codes have been determined by explicit enumeration. The purpose of this paper is to obtain a new upper bound for error detecting and error correcting codes of finite length, without regard to whether the codes are systematic or not.

Definitions and a review of previous pertinent work is given in the next section. One upper bound for the number of code points is obtained in Section III, and a further improvement on this bound is given in Section IV for binary codes. Section V contains a generalization of the previous results to codes using any number of elementary symbols. A number of mathematical details concerning the computation of a certain n-dimensional volume are given in the Appendix.

II. Definitions and Review

Given a set of B elementary symbols $a_1, \cdots, a_B$, an ordered sequence of n symbols (an ordered n-tuple) is called a "letter of length n." If the symbols are independent, then a totality of B^n letters, termed the "alphabet," is possible. The usual case treated occurs when $B = 2$, the two symbols (digits) being 0 and 1; if the letters are of length n then there are 2^n possible letters in the alphabet. Attention here, and in the next two sections, will be confined to the case where $B = 2$.

An error is defined as a change of a "0" to a "1", or vice versa, in any position of a letter. Multiple errors are to be considered as single errors in two or more different positions of a letter.

The occurrence of an error transforms one letter to another; if all the 2^n letters are allowed then there is no way of deciding that an error (or errors) has occurred. It is thus necessary to select a subset of the set of 2^n letters in order to detect the presence of an error. In order to correct an error, the position of the error must be known; the condition is also sufficient in the binary case. Any subset of the set of 2^n letters which permits one to detect or correct errors will be called a set of code points, for geometrical reasons to be discussed later. The subset (or subsets) with the greatest number of members, which will allow one to detect or correct a given number of errors, will be termed an optimal code.

Let m digits of a letter of length n be arbitrary, and let the remaining $k = n - m$ of the digits be fixed functions of the m digits, whose values, however, are either 0 or 1. If all the letters of a set of code points satisfy this condition, the set is called a systematic code.

An upper bound on the number of code points for binary systematic error correcting codes has been obtained by Hamming.[1] Let a maximum of e errors be possible in any letter, then by counting the number of ways in which the k digits can be used to correct 0, 1, 2, $\cdots$, e errors one finds that an upper bound, $\nu_H(n, e)$, on the maximum number of code points, $N_m(n, e)$, is given by

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[1] R. W. Hamming, "Error detecting and error correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–160; April, 1950.

[2] M. J. E. Golay, "Notes on digital coding," Proc. IRE, vol. 37, p. 657; June, 1949.

[3] D. Slepian, "A class of binary signaling alphabets," *Bell Sys. Tech. J.*, vol. 35, pp. 203–234; January, 1956.

[4] A. E. Laemmel, "Efficiency of Noise Reducing Codes," in "Communication Theory," Willis Jackson, Ed., Butterworths Scientific Publications, London, Eng., pp. 111–118; 1953.

[5] D. A. Huffman, "A Method for the Construction of Minimum-Redundancy Codes," *ibid.*, pp. 102–110.

[6] V. I. Siforov, "On Noise Stability of a System with Correcting Codes," in "Supplementary Transactions of the 1956 Symposium on Information Theory," M.I.T., Cambridge, Mass.; September 10–12, 1956.

[7] I. S. Reed, "A class of multiple-error-correcting codes and the decoding scheme," IRE Trans. on Information Theory, no. PGIT-4, pp. 38–49; September, 1954.

[8] E. N. Gilbert, "A comparison of signaling alphabets," *Bell Sys. Tech. J.*, vol. 31, pp. 504–522; May, 1952.

$$N_m(n, e) \leq \frac{2^n}{1 + \binom{n}{1} + \cdots + \binom{n}{e}} = \nu_H(n, e); \qquad (1)$$

is is Hamming's upper bound. A few codes have been und for which $N_m(n, e)$ is equal to the upper bound, but is known also that in some cases $\nu_H(n, e)$ is unattainable. A geometrical model, also introduced by Hamming,[1] akes it possible to discuss codes in a general way, with- t regard to whether they are systematic, or not. Any dered n-tuple of digits may be thought of as a point in dimensional space with each digit representing the alue of a different coordinate of the space; if the digits e binary digits then the point is a vertex of a unit cube. he alphabet of letters represents all the vertices of a nit cube, and the set of code points is a subset of these ertices.

Hamming defines a metric, D, between two vertices as he number of coordinates for which the vertices are ifferent. Note that the Hamming distance, D, is just the quare of the Euclidean distance, d, between vertices. If single error occurs then one letter is changed into nother letter, and $D = 1$ between these two letters. ow if the set of code points is chosen such that every tter is a Hamming distance $D \geq 2$ from every other tter in the set, then a single error in any member of the t will be recognizable, for the letter will be transformed a letter not in the set. The single error will thus be etectable. If the code points are chosen such that $D \geq 3$ r every pair of letters in the set of points then a single rror will be correctable. Codes with a minimum distance $= 4, 5, \cdots$ between points allow detection or correction multiple errors. The determination of optimal codes is hus equivalent to finding the subset (or subsets) with the reatest number of points, which satisfies the minimum istance condition.

A distinction is made between even and odd values of in obtaining upper bounds. Hamming[1] surrounds each ode point by a sphere of radius $R_H = (D - 1)/2 = e$, hen D is odd, and counts the number of vertices within nd on the sphere, thereby obtaining $\nu_H(n, D) = {}_H(n, e = (D - 1)/2)$, as given in (1).

If D is even, then

$$N_m(n - 1, D = 2t - 1) = N_m(n, D = 2t) \qquad (2)$$

derived, t being a positive integer, and used to find pper bounds for even D by noting that $\nu_H(n - 1, e = D - 2)/2 \geq N_m(n - 1, D - 1) = N_m(n, D)$.

III. An Upper Bound on the Number of Code Points (The Hard Sphere Model)

The geometrical model introduced by Hamming, and iscussed in the previous section, may be used to obtain n upper bound, $\nu_1(n, D)$, on the number of code points, or any distance D. This will be done here, and an improve- ent on this bound, $\nu_2(n, D)$, will be obtained in Section V.

Consider a set of code points, whose members are denoted by the symbols r_i, where $r_i = (x_1^{(i)}, \cdots, x_n^{(i)})$; the $x_1^{(i)}$ are either 0 or 1. Imagine a sphere of radius R with center at r_i to surround each code point.

The spheres may touch, but they are not permitted to intersect each other. Thus the distances between code points are greater than or equal to $2R$. (All distances, unless stated otherwise, are Euclidean distances.) One may think of the spheres as "hard," that is, as being composed of an impenetrable, incompressible material whose mass density is unity, throughout the sphere. The interstices between spheres can be considered to be free of matter (mass density zero).

Picture now the spheres of radius R placed at the code points, and the entire structure, supposed rigid, sawed with the spheres fixed in position, along the faces of the unit cube.

The portion of each sphere contained within the unit cube will be called a truncated spherical element. The truncated spherical elements will all have the same volume, because of symmetry. It is convenient to compute the volume of the truncated spherical element whose apex (the point which was originally the center of the sphere, before truncation) is at the origin, $(0, 0, 0, \cdots, 0)$.

The volume of the truncated spherical element, $V(n, R)$ is given by

$$V(n, R) = \int_{\Omega} \cdots \int dx_1 \cdots dx_n, \qquad (3)$$

where Ω is the set of points which is the intersection of the set s_1 and the set c_1, namely $\Omega = s_1 \cap c_1$, with

$$s_1 = \text{the set for which } 0 \leq \sum_{j=1}^{n} x_j^2 \leq R^2,$$

and $c_1 =$ the set for which $0 \leq x_j \leq 1$, $j = 1, 2, \cdots, n$.

It is shown in the Appendix that $V(n, R)$ may be written as

$$V(n, R) = \left(\frac{\pi}{2}\right)^{n/2-1} \int_0^{\infty} \frac{\sin R^2\xi}{\xi} \frac{Re}{\xi^{n/2}} [C(\xi) + iS(\xi)]^n \, d\xi \qquad (4)$$

where $C(\xi)$ and $S(\xi)$ are the Fresnel integrals. The integral on the right-hand side of (4) may be evaluated numerically.

The connection between $V(n, R)$ and $\nu_1(n, R)$, a bound on the maximum number of code points, can be established by comparing the volume (or mass) of the truncated spherical element with the volume (or mass) of the unit cube. One has

$$\nu_1(n, R) = \frac{1}{V(n, R)} \geq N_m(n, R), \qquad (5)$$

where $N_m(n, R)$ is the maximum number of code points.

Various values of $V(n, R)$ have been computed using Illiac, the University of Illinois digital computer. The corresponding values of $\nu_1(n, R)$ are given in Table I together with Hamming's bound $\nu_H(n, e)$ for comparison. It can be seen that $\nu_1 > \nu_H$ for all R.

The comparison with $\nu_H(n, e)$ is effected by relating R to e. Since the Euclidean distance between the centers of the spheres is $2R$, then $R = d/2 = D^{1/2}/2$, or $R^2 = D/4$.

TABLE I

COLLECTED RESULTS

n		R^2 = 0.75, e = 1	R^2 = 1.25, e = 2	R^2 = 1.75, e = 3	R^2 = 2.25, e = 4	R^2 = 2.75, e = 5	R^2 = 3.25, e = 6
7	ν_1	74.2	12.4	4.1			
	ν_H	16	4.4	2			
	ν_2	18.2*	3.1	2.1			
	N_B	16	2	2			
8	ν_1	199.3	25.6	7.1			
	ν_H	28.4	6.9	2.8			
	ν_2	20.3*	5.3	3			
	N_B	20**	4	2			
9	ν_1	56.7	56.2	12.9	4.8		
	ν_H	51.2	11.1	3.9	2		
	ν_2	39.7*	9.2	4.2	2.7		
	N_B	38**	6	2	2		
10	ν_1	1.69×10^3	128.7	25.0	7.6		
	ν_H	93.1	18.3	5.8	2.7		
	ν_2	82.2	16.6	6.1	3.4		
	N_B	68**	12	2	2		
11	ν_1	5.29×10^3	312.5	50.6	13.5	5.5	
	ν_H	170.7	30.6	8.8	3.6	2	
	ν_2	154.8*	26.5*	8.6*	4.3	2.8	
	N_B	128	24	4	2	2	
12	ν_1	1.72×10^4	781.2	106.9	25.5	8.8	
	ν_H	315.1	51.8	13.7	5.2	2.6	
	ν_2	346.8*	46.7*	12.7*	5.7	3.3	
	N_B	256	24	4	2	2	
13	ν_1	5.84×10^4	2.05×10^3	235.8	48.9	15.2	6.3
	ν_H	585.1	89.0	21.7	7.5	3.4	2
	ν_2	806*	85.1*	19.3*	7.2*	4.0*	2.9
	N_B	512	32	8	2	2	2
14	ν_1	2.05×10^5	5.56×10^3	537.6	97.6	26.8	10.0
	ν_H	1.09×10^3	154.6	34.9	11.1	4.7	2.5
	ν_2	1913*	160*	30.1*	9.8*	4.9*	3.3
	N_B	1024	48(?)†	16	4	2	2
15	ν_1	7.44×10^5	1.55×10^4	1.27×10^3	201.6	49.3	16.6
	ν_H	2.05×10^3	270.8	56.9	16.9	6.6	3.3
	ν_2	4656*	309.8*	48.9*	13.8*	6.2*	3.9
	N_B	2048		32	4	2	2
16	ν_1	2.78×10^6	4.47×10^4	3.11×10^3	431.0	94.0	28.5
	ν_H	3.86×10^3	478.4	94.0	26.0	9.5	4.4
	ν_2	1.16×10^4*	617.3*	82.0*	24.2	8.8	4.7
	N_B	2048		32		2	2
17	ν_1	1.07×10^7	1.33×10^5	7.81×10^3	951.5	185.2	50.7
	ν_H	7.32×10^3	851.1	157.2	40.8	13.9	6.0
	ν_2	2.96×10^4*	1264*	140.0*	37.5	12.1	5.9
	N_B	4096		64		4	2

ν_1 = the upper bound obtained from the "hard sphere" model.

ν_2 = the improved upper bound, using the "soft sphere" model.

ν_H = the Hamming bound.

N_B = the best value actually found. These entries have been taken largely from Laemmel,[6] except for entries marked with a double asterisk.

* These entries have been computed using (2). All other ν_2 entries have not used (2).

** These entries are corrections to Laemmel,[6] as given by Golay[2]: they were brought to the author's attention by Dr. B. Elspas.

† This entry is taken directly from Laemmel (6), namely "48 (?)".

urthermore $D = 2e + 1$, in an e error correcting code; us $R = (2e + 1)^{1/2}/2$.

V. Another Upper Bound (The Soft Sphere Model)

It may have been noticed, on examination of Table I hat ν_1 is closer to ν_H for large R (entries on the right-hand de of the table) than for small R. These results occur ecause the interstices between the truncated spherical olume elements occupy a large fraction of the volume of he unit cube when R is small, and a much smaller fraction f the unit cube when R is large. A method of filling up he interstices is needed which will permit one to improve he bound $\nu_1(n, R)$. This is done in this section by adapting echniques developed by Blichfeldt[9] and Rankin[10] in heir studies in the geometry of numbers.

The volume (or mass) of a bounding cube was compared ith the volume (or mass) of a truncated spherical element, n obtaining $\nu_1(n, R)$, and the truncated spherical elements ere assigned a unit weighting function (filled with npenetrable matter of constant mass density unity). onsider however, another system in which the set of riginal spheres, S_1, is replaced by a new set of spheres, $_2$, fixed in the same positions as the spheres, S_1. Let the pheres, S_2, be: 1) of larger radius than the spheres S_1,) filled with matter which is penetrable, but whose variable) mass density is never greater than unity at any oint in space, regardless of how many of the spheres S_2 verlap. The system of spheres, S_2, thus fills the interstices hich were between the spheres, S_1, and one can improve he bound $\nu_1(n, R)$ thereby.

First, the spheres S_2 are taken to be of radius $R_2 = \sqrt{2}R_1$, where R_1 was the radius of the spheres S_1. Next weighting function (mass density function) $\rho(\xi^{(j)})$ is ssigned to each point.

Let $r = (x_1, x_2, \cdots, x_n)$ represent the coordinates of ny point, and let

$$^{(j)} = | r - r^{(j)} | = \{(x_1 - x_1^{(j)})^2 + \cdots + (x_n - x_n^{(j)})^2\}^{1/2}$$

epresent the distance between any point and the jth ode point (the center of the jth sphere). One defines $_1(\xi^{(j)})$ by

$$\begin{aligned} \rho_1 &= 1, \qquad 0 \le \xi^{(j)} \le (\sqrt{2} - 1)R_2 \\ &= 1 - \frac{\xi^{(j)2}}{R^2}, \qquad (\sqrt{2} - 1)R_2 \le \xi^{(j)} \le R_2 \qquad (6) \\ &= 0, \qquad \xi^{(j)} \ge R_2. \end{aligned}$$

This is one of the weighting functions used by Blichfeldt[9] who has shown that the total mass density t any point is less than or equal to unity, *i.e.*,

$$\sum_j \rho_1(\xi^{(j)}) \le 1.$$

Two other weighting functions have been found to be useful. $\rho_2(\xi^{(j)})$ is given by

$$\begin{aligned} \rho_2 &= \rho_1, \qquad 0 \le \xi^{(j)} \le \frac{R_2}{\sqrt{2}} \\ &= \frac{1}{2} - \frac{(n-1)\log \xi^{(j)}}{2(n+1)\log\left(\frac{nR_2}{n+1}\right)}, \qquad \frac{R_2}{\sqrt{2}} \le \xi^{(j)} < R_2 \qquad (7) \\ &= 0, \qquad \xi^{(j)} \ge R_2 \end{aligned}$$

Finally, ρ_3, particularly useful for large n, is defined by[11]

$$\begin{aligned} \rho_3 &= 1, \qquad 0 \le \xi^{(j)} \le (\sqrt{2} - 1)R_2 \\ &= 1 - \left(\frac{\xi^{(j)}}{R_2}\right)^2, \\ &\qquad \cdot(\sqrt{2} - 1)R_2 \le \xi^{(j)} \le \left(\frac{n}{n+\alpha}\right)^{1/2} R_2 \\ &= \frac{1}{n+1} - \left(1 - \frac{1+\alpha}{2n}\right)\log\frac{(n+1)\xi^{(j)2}}{nR_2^2}, \\ &\qquad \cdot\left(\frac{n}{n+\alpha}\right)^{1/2} R_2 \le \xi^{(j)} \le R_2 \\ &= 0, \qquad \xi^{(j)} > R_2, \end{aligned} \qquad (8)$$

where $\alpha = 2.51286 \cdots$, is the nonzero root of

$$e^{\alpha/2} = 1 + \alpha.$$

Both $\sum_j \rho_2(\xi^{(j)})$ and $\sum_j \rho_3(\xi^{(j)})$ are less than or equal to one, for all points; proofs can be found in the references.

Finally, the system of spheres, S_2, is truncated, as before, along the faces of the unit cube. The total mass, M_T, of all of the overlapping truncated spherical elements will be given by

$$\begin{aligned} M_T &= \sum_j \int_{c_1} \cdots \int \rho(\xi^{(j)})\, dx_1 \cdots dx_n \\ &\ge N_m(n, R) \int_{\psi} \cdots \int \rho(r)\, dx_1 \cdots dx_n \\ &= N_m(n, R) M(n, R_2), \end{aligned} \qquad (9)$$

where ψ is the set of points: $s_2 \cap c_1$, s_2: $0 \le \sum_{k=1}^n x_k^2 \le R_2^2$ and c_1 is the unit cube. $\rho(r)$ is the mass density distribution function for a single spherical element (without overlap), r represents the distance from the origin, and is used in place of $\xi^{(j)}$ in (6)–(8), and $M(n, R_2)$ is the mass of a single spherical element. $N_m(n, R)$ is, again, the maximum number of code points.

Let the unit cube be filled with matter of mass density unity. Then an upper bound on $N_m(n, R)$ is found from

$$1 \ge M_T \ge N_m M(n, R_2)$$

[9] H. F. Blichfeldt, "The minimum value of quadratic forms, and he closest packing of spheres," *Math. Ann.*, vol. 101, pp. 605–608; 929.

[10] R. A. Rankin, "On the closest packing of spheres in n di- ensions," *Annals of Math.*, vol. 48, pp. 1062–1081; October, 1947.

[11] This definition for $\rho_3(\xi^{(j)})$, chosen for its relative simplicity, differs from that given by Rankin, but not in a significant way. The major contributions to the sphere's volume and mass come from the outermost portions of the sphere, where the two definitions coincide.

or

$$\nu_2(n, R) = \frac{1}{M(n, R_2)} \geq N_m(n, R). \tag{10}$$

Various expressions for $M(n, R_2)$ may be obtained, depending on the value of R_2, and the weighting function used.

If $R_2 \leq 1$ then $M(n, R_2)$ can be given exactly. Note that ψ is the 2^{-n}th part of the sphere of radius R_2, hence

$$\begin{aligned} M(n, R_2) &= \int_{\psi} \cdots \int \rho(r)\, dx_1 \cdots dx_n \\ &= \frac{\int_{S_2} \cdots \int \rho(r)\, dx_1 \cdots dx_n}{\int_{S_2} \cdots \int dx_1 \cdots dx_n} V(n, R_2) \\ &= \langle \rho \rangle V(n, R_2). \end{aligned} \tag{11}$$

If $\rho_2(r)$ is used then

$$\langle \rho_2 \rangle = \frac{2 + n(\sqrt{2} - 1)^{n+2}}{n + 2}. \tag{12}$$

A slight improvement on $\langle \rho_2 \rangle$ is possible for large n, by using $\langle \rho_3 \rangle$. One finds that

$$\langle \rho_3 \rangle = \frac{2}{n}\left\{1 - \beta/n + 0\left(\frac{1}{n^2}\right)\right\}, \tag{13}$$

where $\beta = 1.91288 \cdots$.

If $R_2 > 1$ then the only exact expression found for $M(n, R_2)$ has been obtained by using $\rho_1(r)$. It is

$$\begin{aligned} M(n, \sqrt{2}R_1) = M(n, R_2) &= V(n, \sqrt{2}R) - \frac{n}{\pi R_2^2}\left(\frac{2}{\pi}\right)^{n-1/2} \\ &\cdot \int_0^{\infty} \frac{\sin \xi b}{\xi} \frac{\sin (n-1)\phi + \xi(1-a)}{\xi} \left(\frac{A}{\sqrt{\xi}}\right)^{n-1} d\xi \\ &+ \frac{n}{\pi R_2^2}\left(\frac{2}{\pi}\right)^{n/2} \\ &\cdot \int_0^{\infty} \frac{\sin \xi b}{\xi} \frac{\sin (n\phi - \xi a)}{\xi} \left(\frac{A}{\sqrt{\xi}}\right)^{n-1} d\xi, \end{aligned} \tag{14}$$

with $a = R_2^2\sqrt{2}\,(\sqrt{2} - 1)$ and $b = R_2^2\,(\sqrt{2} - 1)$. No proof of (14) is given here; (14) was obtained by techniques similar to those used in evaluating $V(n, R)$, as shown in the Appendix.

The above formulation for $M(n, R_2)$ suffers from the disadvantage that only the crudest of the three weighting functions has been employed. The other weighting functions were used by integrating (9) by parts, using the previously computed tables of values of $V(n, R)$. Larger values of $M(n, R_2)$ were obtained for small n and R_2 using $\rho_2(r)$, rather than $\rho_3(r)$, whereas the situation was reversed when n and R_2 were large.

A set of values of $\nu_2(n, R)$, $\nu_H(n, R)$, and the maximum number of code points known, are given in Table I. The values of $\nu_2(n, R)$ were often improved by using (2); the best values are listed.

The bounds obtained here, $\nu_2(n, R)$, are smaller than $\nu_H(n, R)$ for values of R roughly in the central columns of Table I. These results might have been anticipated. When R is small then ν_H can sometimes be attained, thus no improvement over ν_H is possible for those cases. Furthermore ν_H is always fairly close to the actual values found for small R; hence little improvement is likely, in general. When R is large, namely when $D > \frac{2}{3}n$ or $R > (n/6)^{1/2}$, then it is known that $N_m(n, R) = 2$, and $N_m(n, \sqrt{n/6}) = 4$. However $\nu_H(n, R)$ is not for from either of these values for appropriately large R. Hence slight improvement on ν_H should be expected even for the most refined $\rho(r)$. It is only for intermediate values of R that $\nu_2 < \nu_H$ might, and does, occur with regularity.

V. Generalization to Any Base

The discussion in the previous sections has been limited, almost entirely, to binary codes. It is possible to extend the results given above to error detecting codes using B elementary symbols, if a slightly artificial quantitative definition of multiple error is adopted.

Consider a code using the elementary symbols 0, 1, 2. If the symbol 2, at a fixed position in a letter, is transformed to a 1, then it is natural to say that a single error has occurred. Suppose, however, that $2 \rightarrow 0$; is this to be considered a single or a multiple error? It is convenient, and, in part justifiable intuitively, to define the change $2 \rightarrow 0$ as a double error. In general, given two letters of the alphabet $P_i = (x_1^{(i)}, \cdots, x_n^{(i)})$ and $P_k = (x_1^{(k)}, \cdots, x_n^{(k)})$ with each x_j one of the B elementary symbols 0, 1, $\cdots$, $B - 1$. Then the transformation $P_i \rightarrow P_k$ will be defined as an error of multiplicity e whenever

$$\sum_{i=1}^{n} \left| x_i^{(i)} - x_i^{(k)} \right| = e.$$

In the binary case if e is given then the Euclidean distance, d, between two points is $d^2 = e$. This is no longer true when $B > 2$, since for given e, d is not determined uniquely. A smallest (integral valued) d^2 exists, however, for each (integral valued) e; this smallest d, $d_{\min}(e)$, will be used as the Euclidean distance associated with given e.

It should also be noted that error correction is not possible, in general, if the location and the multiplicity of the error is specified. Thus if $B = 5$, and if it is known that a particular symbol, a 2, is in error and that two errors have occurred, then either a 0 or a 4 was the original symbol. Ambiguity still exists, and one needs to know the sign of the error in order to correct the error. The bounds found in this section are thus bounds on the maximum number of code points for error detecting codes.

The possible code points are the lattice points of a cube of side $b = B - 1$, where a lattice point is a point with integral coordinates. All the previous ideas may now be applied. In particular, the volume of the cube is b^n, each code point is surrounded by a sphere of radius R_1, where

$$R_1 = \frac{d_{\min}(e)}{2}, \tag{15}$$

nd the spheres are truncated along the faces of the cube. imilarly, the spheres of radius R_1 may be replaced by pheres of radius $R_2 = \sqrt{2}R_1$, and the density functions sed, as before.

A simple scaling argument allows one to use the previous esults.

Let $V(n, R_1, b)$ be the volume of the truncated spherical lement, when the cube is of side b. Then

$$V(n, R_1, b) = \int \cdots \int_{\Omega^1} dx_1 \cdots dx_n \tag{16}$$

with $\Omega^1 = s \cap c_b$, where s: $0 \leq \sum_{j=1}^n x_j^2 \leq R^2$, c_b: $0 \leq x_j \leq b$, $j = 1, \cdots, n$.

Let $y_j = \mu x_j$, with $0 \leq y_j \leq 1$, $j = 1, \cdots, n$, and $\mu = 1/b$, then

$$\begin{aligned} V(n, R, b) &= \int \cdots \int_{\Omega^1} dx_1 \cdots dx_n \\ &= \frac{1}{\mu^n} \int \cdots \int dy_1 \cdots dy_n = b^n V(n, R/b, 1) \\ &= b^n V(n, R/b), \end{aligned} \tag{17}$$

using the previous notation.

Hence

$$\nu_1(n, R, b) = \frac{b^n}{V(n, R, b)} = \frac{1}{V(n, R/b)}, \tag{18}$$

and similarly

$$\nu_2(n, R, b) = \frac{1}{M(n, R_2/b)}. \tag{19}$$

Appendix

On the Evaluation of $V(n, R)$

The volume of the truncated spherical element, $V(n, R)$ was given by (3), namely

$$V(n, R) = \int \cdots \int_{\Omega} dx_1 \cdots dx_n \tag{20}$$

where

$$\Omega : s_1 \cap c_1; s_1 : 0 \leq \sum_{j=1}^n x_j^2 \leq R^2;$$

$$c_1 : 0 \leq x_j \leq 1, \qquad j = 1, 2, \cdots n.$$

The multiple integral in (20) is difficult to compute, as one must determine the regions of intersection of the unit cube and the sphere and then integrate over the regions. These difficulties may be avoided by using the discontinuous Dirichlet factor, δ, to sort out the spherical region automatically.

Let

$$\delta = \frac{2}{\pi} \int_0^\infty \frac{\sin \xi R^2}{\xi} \cos\left(\xi \sum_{j=1}^n x_j^2\right) d\xi, \tag{21}$$

then $\delta = 1$ whenever

$$0 < \sum_{j=1}^n x_j^2 < R^2, \quad \text{and} \quad \delta = 0 \quad \text{otherwise;}$$

δ therefore selects the 2^{-n}th part of the sphere.

Consequently $V(n, R)$ can be expressed by

$$V(n, R) = \frac{2}{\pi} \int_0^\infty \int_0^1 \cdots \int_0^1 \frac{\sin \xi R^2}{\xi} \cdot \cos\left(\xi \sum_{j=1}^n x_j^2\right) dx_1 \cdots dx_n \, d\xi \tag{22}$$

$$= \frac{2}{\pi} Re \int_0^\infty \int_0^1 \cdots \int_0^1 \frac{\sin \xi R^2}{\xi} \cdot \exp\left(i\xi \sum_{j=1}^n x_j^2\right) dx_1 \cdots dx_n \, d\xi \tag{23}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin \xi R^2}{\xi} \cdot Re \left\{\int_0^1 \exp(i\xi x^2)\, dx\right\}^n d\xi. \tag{24}$$

Since

$$\int_0^1 \exp(i\xi x^2)\, dx = \left(\frac{\pi}{2\xi}\right)^{1/2} \int_0^u \exp\left(i \frac{\pi}{2} t^2\right) dt, \tag{25}$$

where $u = (2\xi/\pi)^{1/2}$, and since

$$\int_0^u \exp\left(i \frac{\pi}{2} t^2\right) dt = C(\xi) + iS(\xi) = \frac{1}{2} \int_0^\xi J_{-1/2}(\eta)\, d\eta + \frac{i}{2} \int_0^\xi J_{1/2}(\eta)\, d\eta, \tag{26}$$

with $C(\xi)$ and $S(\xi)$ the Fresnel integrals, one has that

$$V(n, R) = \left(\frac{\pi}{2}\right)^{n/2-1} \int_0^\infty \frac{\sin \xi R^2}{\xi} Re \left\{\frac{C(\xi) + iS(\xi)}{\xi^{1/2}}\right\}^n d\xi, \tag{27}$$

which is (4).

It is convenient in determining $V(n, R)$ numerically to let

$$C(\xi) + iS(\xi) = Ae^{i\phi},$$

then

$$V(n, R) = \left(\frac{\pi}{2}\right)^{n/2-1} \int_0^\infty \frac{\sin \xi R^2}{\xi} \frac{A^n \cos n\phi}{\xi^{n/2}} d\xi. \tag{28}$$

A truncated approximation to (28) was used for the computations. The Fresnel integrals were generated from their series expansions, and the functions $A/\sqrt{\xi}$ and ϕ computed, within Illiac. These functions were then used, as indicated in (28), to obtain $V(n, R)$.

Two arithmetic checks on the accuracy of the numerical integrations are available immediately.

If $R \leq 1$, then (28) should represent the 2^{-n}th part of a sphere, namely

$$V(n, R) = \frac{2^{-n}\pi^{n/2}R^n}{\Gamma(1 + n/2)}, \tag{29}$$

where $\Gamma(x)$ denotes the Gamma function.

Eq. (29) has been used to check the numerical integrations. The results were accurate to 10 decimal places when $R = 1$ (for $n \leq 20$).

The volume of the truncated spherical element is the volume of the unit cube, whenever $R^2 \geq n$. This affords another check.

The numerical integrations were carried out to $R^2 = n + 3$ in order to test the accuracy of the methods, and were found to be correct to at least 8 decimal places (for $n \leq 20$).

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The Probability Density of the Output of an RC Filter When the Input Is a Binary Random Process*

J. A. McFADDEN†

Summary—**A new method is given for obtaining the probability density of the output of an RC filter when the input is a stationary binary random process. The axis-crossing intervals of the input are assumed to be statistically independent and identically distributed, but with an arbitrary density function. The method involves a linear integral equation which can be reduced by Laplace transforms. A new family of solutions is given which includes two previously known cases: the random square wave of Poisson type, and the periodic square wave with random time origin. The general result of this family is given in terms of tabulated functions. For other solutions, a recursive technique may be necessary.**

Introduction

THIS PAPER treats the probability density of the output of an RC filter when the input is a binary random process. Previously, Wonham and Fuller[1] gave the result for the case in which the input is a random square wave of Poisson type, *i.e.*, the random telegraphic signal. For the same input, McFadden[2] has obtained the solution when the system is an ideal integrator with finite memory. Again, for the random square wave of Poisson type, but with an arbitrary linear system, methods have been given by McFadden[3] and by Wonham.[4]

In the present paper, the system is again specialized to the case of the RC low-pass filter, but the input is generalized. The input considered here is a stationary binary random process, the axis-crossing intervals of which are statistically independent and identically distributed. The distribution of the lengths of these intervals is arbitrary. (It should be noted that this input is *not* a Markov process.)

Derivation of the Basic Equations

The differential equation governing the given system is the following:

$$x(t) = y(t) + T\frac{dy(t)}{dt}, \tag{1}$$

where $x(t)$ is the input, $y(t)$ is the output, t is time, and $T = RC$, the time constant of the system.

Let the input assume only the values ± 1, with either value being equally probable. Let t_0 be a particular instant at which $x(t)$ changes from negative to positive, and let t_1 be the next instant at which $x(t)$ changes from positive to negative. Let $y(t_0) = y_0$ and $y(t_1) = y_1$; $y_0 \leq y_1$, since $x(t)$ is positive during this interval. The probability density of the maximum amplitudes is

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[1] W. M. Wonham and A. T. Fuller, "Probability densities of the smoothed 'random telegraph signal'," *J. Electronics and Control*, vol. 4, pp. 567–576; June, 1958.

[2] J. A. McFadden, "The probability density of the output of a filter when the input is a random telegraphic signal," 1959 IRE National Convention Record, pt. 4, pp. 164–169.

[3] J. A. McFadden, "The probability density of the output of a filter when the input is a random telegraphic signal—differential-equation method," IRE Trans. on Information Theory, vol. IT-5, pp. 228–233; May, 1959.

[4] W. M. Wonham, private communication. To be published in *J. Electronics and Control.*

$Q_+(y_1)$, and the probability density of the minimum amplitudes is $Q_-(y_0)$. By symmetry,

$$Q_+(y_1) = Q_-(-y_1). \tag{2}$$

From the solution of the differential equation (1), the amplitudes y_0 and y_1 are related by

$$y_1 = 1 - (1 - y_0)e^{-\tau/T}, \tag{3}$$

where $\tau = t_1 - t_0$, which is the length of the appropriate axis-crossing interval of the input. The probability density of these intervals is $P_0(\tau)$, which is assumed to be known.

Now, if successive axis-crossing intervals of the input are statistically independent, y_0 and τ will be independent, but y_1 and τ will be dependent. The joint probability density of y_0 and τ is $Q_-(y_0)P_0(\tau)$. By (3), the variable y_0 may be eliminated, and it follows that the joint density of y_1 and τ is

$$e^{\tau/T}Q_-[1 - (1 - y_1)e^{\tau/T}]P_0(\tau). \tag{4}$$

Integration over the possible values of τ will yield the marginal density $Q_+(y_1)$. The upper limit for this integration is obtained from (3) when $y_0 = -1$. The result for $Q_+(y_1)$ is the following:

$$Q_+(y_1) = \int_0^{T \log [2/(1-y_1)]} e^{\tau/T}P_0(\tau) \cdot Q_-[1 - (1 - y_1)e^{\tau/T}]\, d\tau. \tag{5}$$

Thus, if $P_0(\tau)$ is known, the probability densities of the maxima and minima can be obtained, in principle, from the solution of this integral equation, under the restriction (2).

The next step is to calculate the probability density of the output at an arbitrary instant in terms of the densities of the maxima and minima. Let t be an arbitrary instant between t_0 and t_1, *i.e.*, during the period in which the input is positive. Let the amplitude of the output at that instant t be denoted simply by y. The probability density of this amplitude is denoted by $P_+(y)$. If $l = t - t_0$, then y is related to the minimum amplitude y_0 by the relation,

$$y = 1 - (1 - y_0)e^{-l/T}. \tag{6}$$

Then, by an argument similar to the previous one, the probability density of $y(t)$ is related to the density of minimum amplitudes by the following integral:

$$P_+(y) = \int_0^{T \log [2/(1-y)]} e^{l/T}Q_0(l)Q_-[1 - (1 - y)e^{l/T}]\, dl, \tag{7}$$

where $Q_0(l)$ is the probability density of the time difference $t - t_0$, where t is fixed but t_0 is random.

It appears, therefore, that if $Q_-(y_0)$ can be determined so as to satisfy (2) and (5), then the density $P_+(y)$ of the amplitude at an arbitrary instant (while x is positive) can be determined from (7), provided $Q_0(l)$ is known. There remains the task of calculating $Q_0(l)$ in terms of the density of axis-crossing intervals.

The probability that the first axis crossing before t occurred between t_0 and $t_0 + dl$ is $Q_0(l)dl$, since $l = t - t_0$. The same event may also be described as follows: the probability that an axis crossing occurs between t_0 and $t_0 + dl$ and that the next axis crossing occurs after time t is[5]

$$\beta\, dl \int_l^\infty P_0(\tau)\, d\tau, \tag{8}$$

where β is the expected number of crossings per unit time in the input. Then, by equating these two probabilities,

$$Q_0(l) = \beta \int_l^\infty P_0(\tau)\, d\tau. \tag{9}$$

The problem is completely formulated for the determination of $P_+(y)$ in terms of a given function $P_0(\tau)$. By symmetry, the probability density of the amplitude at an arbitrary time for which the input is *negative* is

$$P_-(y) = P_+(-y). \tag{10}$$

Finally, the probability density $P(y)$ of the amplitude of the output at any given instant, regardless of the sign of the input, is the mean value.

$$P(y) = \tfrac{1}{2}[P_+(y) + P_-(y)]. \tag{11}$$

Reduction by Laplace Transforms

The problem will now be reduced to a simpler one by means of Laplace transforms. Let the amplitude variables be transformed as follows:

$$\begin{aligned} y_1 &= 1 - 2e^{-\lambda_1/T} \\ y &= 1 - 2e^{-\lambda/T}. \end{aligned} \tag{12}$$

Furthermore, let the probability density functions be redefined as follows:

$$\begin{aligned} Q_+(y) &= H_1(\lambda); \qquad Q_-(y) = H_2(\lambda); \\ P_+(y) &= G_1(\lambda). \end{aligned} \tag{13}$$

Then the integrals in (5) and (7) become convolution integrals and the Laplace transform may be employed. Let

$$\begin{aligned} \mathfrak{L}\{H_1(\lambda)\} &= h_1(s); \qquad \mathfrak{L}\{H_2(\lambda)\} = h_2(s); \\ \mathfrak{L}\{G_1(\lambda)\} &= g_1(s); \\ \mathfrak{L}\{P_0(\lambda)\} &= p_0(s); \qquad \mathfrak{L}\{Q_0(\lambda)\} = q_0(s); \end{aligned} \tag{14}$$

where

$$\mathfrak{L}\{F(\lambda)\} = \int_0^\infty e^{-\lambda s}F(\lambda)\, d\lambda. \tag{15}$$

[5] J. A. McFadden, "The axis-crossing intervals of random functions II," IRE Trans. on Information Theory, vol. IT-4, pp. 14-24, (76); March, 1958.

Under these transformations, (5) and (7) become the following, respectively:

$$h_1(s) = p_0\left(s - \frac{1}{T}\right)h_2(s), \tag{16}$$

$$g_1(s) = q_0\left(s - \frac{1}{T}\right)h_2(s). \tag{17}$$

Furthermore, by an elementary calculation, the transformation of (9) yields the result

$$q_0(s) = \frac{\beta}{s}\,[1 - p_0(s)]. \tag{18}$$

The constant β may also be calculated from $p_0(s)$. Since $1/\beta$ is the mean axis-crossing interval,[6]

$$\beta = -1/p_0'(0). \tag{19}$$

The expression for $q_0(s)$ may be substituted into (17); then with the aid of (16) a simple relation follows for the function $g_1(s)$ in terms of $h_1(s)$ and $h_2(s)$. The result is the following, where $\kappa = \beta T$:

$$g_1(s) = \frac{\kappa}{sT - 1}\,[h_2(s) - h_1(s)]. \tag{20}$$

Thus, if $h_1(s)$ and $h_2(s)$ are known, $g_1(s)$ follows from (20). Then the inverse transformation yields $P_+(y)$, and by (10) and (11), the solution can be completed. The principal problem remaining is that of determining $h_1(s)$ and $h_2(s)$.

If $p_0(s)$ is known,[7] one relation is provided by (16). The other relation must be obtained from (2), the requirement of symmetry. Under the change of variables (12) and (13), (2) becomes the following:

$$H_1(\lambda) = H_2[-T \log (1 - e^{-\lambda/T})] \tag{21}$$

or

$$H_2(\lambda) = H_1[-T \log (1 - e^{-\lambda/T})]. \tag{22}$$

If these equations are subjected to the Laplace transformation, the resulting integrals may be reduced by a change of variable and by the binomial expansion. Then it follows that

$$h_1(s) = \sum_{k=0}^{\infty} \frac{(1 - sT)_k}{k!}\, h_2\left(\frac{k+1}{T}\right); \tag{23}$$

$$h_2(s) = \sum_{k=0}^{\infty} \frac{(1 - sT)_k}{k!}\, h_1\left(\frac{k+1}{T}\right); \tag{24}$$

where $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ and $(\alpha)_0 = 1$.

It appears, therefore, that the determination of $h_1(s)$ and $h_2(s)$ requires a solution of (16) and either (23) or (24).

[6] *Ibid.*, pp. 17–18.

[7] $p_0(s)$ is closely related to the autocorrelation function and the spectral density of the input. See (30) of McFadden.[5] Also, J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N. Y., p. 45 (36b); 1950.

General Method of Solution

A general method will now be given by which $h_1(s)$ and $h_2(s)$ may be determined when $p_0(s)$ is specified. Inspection of (23) and (24) reveals that $h_1(s)$ and $h_2(s)$ will be known for all values of s if they are known for the discrete values $s = (k + 1)/T$, where k is a non-negative integer. The functions can be determined at these points by a recursive method, since the series terminate when $(1 - sT)$ is a negative integer.

Suppose $h_1[(k + 1)/T]$ is known for the values $k = 0, 1, 2, \cdots, n - 1$. Then let $s = (n + 1)/T$ in (24). The last nonvanishing term in the series can be expressed in terms of $h_2[(n + 1)/T]$ by means of (16). Then the solution for $h_2[(n + 1)/T]$ is the following:

$$h_2\left(\frac{n+1}{T}\right) = \frac{\sum_{k=0}^{n-1} \frac{(-n)_k}{k!}\, h_1\left(\frac{k+1}{T}\right)}{1 - (-1)^n p_0(n/T)}. \tag{25}$$

Then, $h_1[(n + 1)/T]$ is obtained by (16) and the process can be repeated for the next integral value of n. The initial values $(n = 0)$ are obtained from the normalization of Q_+ and Q_-. For all functions $p_0(s)$,

$$h_1(1/T) = h_2(1/T) = T/2. \tag{26}$$

If a general expression can be obtained for $h_1[(n + 1)/T]$ and the series (24) summed in closed form, $h_1(s)$ can be calculated from (16), and $g_1(s)$ follows from (20). The chief difficulty arises in the determination of the nth term of the series.

This section will be concluded with the derivation of an identity in $g_1(s)$. If (23) and (24) are substituted into (20) and the series combined, the terms of the series can be recognized as special cases of (20) for integral values of sT. The final result is the following:

$$g_1(s) = \sum_{k=0}^{\infty} \frac{(2 - sT)_k}{k!}\, g_1\left(\frac{k+2}{T}\right). \tag{27}$$

This series can be considered as the Gregory-Newton interpolation formula for a permissible solution of the present problem.[8]

Note: The coefficients $h_1[(k + 1)/T]$ and $h_2[(k + 1)/T]$ are proportional to the kth moments of $(1 - y_1)$ and $(1 - y_0)$, respectively. Furthermore, $g_1[(k + 1)/T]$ is proportional to the kth conditional moment of $(1 - y)$, given that x is positive. (The constant of proportionality is $T/2^{k+1}$.) Thus, the recursive method given above is essentially a determination of the successive moments of the output.

If, in a particular problem, a finite number of moments of the output is desired (but beyond the mean square), then the present method is recommended over the standard one. The standard method requires a knowledge of the higher product moments of the input, and these

[8] This observation is due to Prof. I. Marx. See E. T. Whittaker and G. Robinson, "The Calculus of Observations," Blackie and Son, Ltd., London and Glasgow, p. 10; 1924.

are not generally known for binary processes.[9] It is far simpler to make assumptions about $P_0(\tau)$ than to work with the higher product moments.

Particular Solutions

The simplest method for discovering new solutions of the given problem is a semi-inverse one, rather than the direct recursive method given above. If $Q_+(y)$ and $Q_-(y)$ are chosen so as to obey (2), and properly normalized over the range $(-1, 1)$, the other functions will all follow directly if the transforms and inverse transforms can be calculated.

Suppose that the probability densities of maxima and minima are those belonging to beta distributions, shifted to the proper range.[10]

$$\begin{aligned} Q_+(y) &= \frac{(1-y)^a(1+y)^b}{2^{a+b+1}B(a+1,\,b+1)}, \\ Q_-(y) &= \frac{(1-y)^b(1+y)^a}{2^{a+b+1}B(a+1,\,b+1)}, \end{aligned} \qquad |y| \leq 1; \tag{28}$$

where each density is zero outside the prescribed range, and $-1 < a < b$.

After the change of variables (12) and (13), the corresponding functions H_1 and H_2 are the following:

$$\begin{aligned} H_1(\lambda) &= \frac{e^{-a\lambda/T}(1-e^{-\lambda/T})^b}{2B(a+1,\,b+1)}, \\ H_2(\lambda) &= \frac{e^{-b\lambda/T}(1-e^{-\lambda/T})^a}{2B(a+1,\,b+1)}, \end{aligned} \qquad 0 \leq \lambda < \infty. \tag{29}$$

The Laplace transforms of these functions may be obtained by a straightforward integration as in (15):

$$\begin{aligned} h_1(s) &= \frac{T}{2}\frac{B(sT+a,\,b+1)}{B(a+1,\,b+1)}, \\ h_2(s) &= \frac{T}{2}\frac{B(sT+b,\,a+1)}{B(a+1,\,b+1)}. \end{aligned} \tag{30}$$

These transforms may be substituted into (16), which may then be solved for the transform of the axis-crossing interval density:

$$p_0(s) = \frac{\Gamma(b+1)}{\Gamma(a+1)}\frac{\Gamma(sT+a+1)}{\Gamma(sT+b+1)}. \tag{31}$$

The interval density can then be obtained with the aid of tables of Laplace transforms:[11]

$$P_0(\tau) = \frac{1}{B(a+1,\,b-a)T}\,e^{-(a+1)\tau/T}(1-e^{-\tau/T})^{b-a-1}. \tag{32}$$

The next step is the determination of the probability density of the output. From (20), $G_1(\lambda)$ can be written as the convolution of the inverse transforms of the two factors of the right member. One of these inverse transforms is elementary, and the other is obtained from the difference of the two functions (29). Then, by the change of variables (12) and (13), the following result appears for the probability density of the output at an arbitrary instant while the input is positive:

$$\begin{aligned} P_+(y) &= \frac{\kappa}{(1-y)B(a+1,\,b+1)} \\ &\quad \cdot \int_{(1/2)(1-y)}^{1} [\xi^b(1-\xi)^a - \xi^a(1-\xi)^b]\,d\xi, \qquad |y| \leq 1; \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{33}$$

Finally, the probability density of the output at an arbitrary instant, regardless of the sign of the input, may be obtained from (10) and (11). By a simple manipulation, the result may be expressed in terms of incomplete beta functions:

$$\begin{aligned} P(y) &= \frac{\kappa}{1-y^2}\,[1 - I_{(1/2)(1+y)}(b+1,\,a+1) \\ &\quad - I_{(1/2)(1-y)}(b+1,\,a+1)], \qquad |y| \leq 1; \\ &= 0 \qquad \text{elsewhere;} \end{aligned} \tag{34}$$

where

$$I_x(p,\,q) = \frac{\int_0^x \xi^{p-1}(1-\xi)^{q-1}\,d\xi}{B(p,\,q)}. \tag{35}$$

I_x is a tabulated function.[12]

The constant κ $(= \beta T)$ is related to a and b as follows: β can be obtained from (31) by means of (19). The result is in terms of the logarithmic derivative of the gamma function:[13]

$$\frac{1}{\kappa} = \psi(b+1) - \psi(a+1), \tag{36}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

The above solution includes two previously known cases. If $b = a + 1$, then $\kappa = b$ and (32) becomes a simple exponential, which is characteristic of the axis-crossing intervals of a random square wave of Poisson type. The density $P(y)$ in (34) becomes the particular solution of Wonham and Fuller[1] for this input.

If a and b both become infinite, but in such a way that b/a remains finite, then $P_0(\tau)$ becomes a δ-function:

$$P_0(\tau) \sim \delta\left(\tau - T\log\frac{b}{a}\right). \tag{37}$$

[9] J. A. McFadden, "The fourth product moment of infinitely clipped noise," IRE Trans. on Information Theory, vol. IT-4, pp. 159–162; December, 1958.

[10] H. Cramér, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., p. 243; 1946.

[11] Bateman Manuscript Project, "Tables of Integral Transforms," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 261, (1), p. 129, (5); 1954.

[12] K. Pearson, "Tables of the Incomplete Beta Function," Cambridge University Press, Cambridge, England; 1934.

[13] Bateman Manuscript Project, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 15; 1953.

Thus, the input becomes a periodic square wave with random time origin. If the input is to be stationary, the time origin must be uniformly distributed over a complete period.

The probability density of the output of an *RC* low-pass filter, when the input is such a periodic square wave, has been calculated by McFadden.[14] The solution (34) becomes identical to the previous one in the limit, where $1/\kappa \sim \log (b/a)$.

For other values of a and b, $P(y)$ may be calculated from the tables.[12] For $a = 5$ and $b = 40$, the probability density of axis-crossing intervals of the input is shown in Fig. 1. This curve is suggestive of an infinitely clipped, narrow-band Gaussian signal. (In reality, however, such a signal would display a strong dependence between successive axis-crossing intervals,[15] and this dependence has been neglected here.) For this case, $\kappa = \beta T = 0.50125$.

For the same case, the solution for $P(y)$ is shown in Fig. 2. This shape approaches the limiting form of the periodic case,[14] for which $P(y)$ is concave upward between upper and lower bounds (these bounds lying inside of $y = \pm 1$) and zero elsewhere.

Another particular solution, not included under (34), can be determined from a different assumption on Q_+ and Q_-. Suppose

$$\begin{aligned} Q_+(y) &= 1, \qquad 0 \le y \le 1; \\ &= 0 \qquad \text{elsewhere}; \\ Q_-(y) &= 1, \qquad -1 \le y \le 0; \\ &= 0 \qquad \text{elsewhere}. \end{aligned} \tag{38}$$

Then, by the same procedure as in the previous example,

$$\left.\begin{aligned} H_1(\lambda) &= 1, \qquad T \log 2 \le \lambda < \infty; \\ &= 0 \qquad \text{elsewhere}; \\ H_2(\lambda) &= 1, \qquad 0 \le \lambda \le T \log 2; \\ &= 0 \qquad \text{elsewhere}. \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} h_1(s) &= 2^{-sT}/s, \\ h_2(s) &= (1 - 2^{-sT})/s; \end{aligned}\right\} \tag{40}$$

$$p_0(s) = 1/(2^{1+sT} - 1); \tag{41}$$

$$P_0(\tau) = \sum_{k=1}^{\infty} 2^{-k}\, \delta(\tau - kT \log 2). \tag{42}$$

This is the axis-crossing interval density for a symmetric coin-toss square wave with a random time origin, and the elementary pulse width equal to $(T \log 2)$.[16] The densities $P_+(y)$ and $P(y)$ may also be obtained as before:

$$\begin{aligned} P_+(y) &= \kappa \frac{1+y}{1-y}, \qquad -1 \le y \le 0; \\ &= \kappa, \qquad 0 \le y \le 1. \end{aligned} \tag{43}$$

$$\begin{aligned} P(y) &= \frac{\kappa}{1 + |y|}, \qquad |y| \le 1; \\ &= 0 \qquad \text{elsewhere}; \end{aligned} \tag{44}$$

where $\kappa = (2 \log 2)^{-1}$. This solution was discovered previously by Wonham, who used a different method.[17] No solution has yet been obtained in closed form for the coin-toss input with an *arbitrary* ratio of time constants.



Fig. 1—Probability density of axis-crossing intervals of the input as given by (32), when $a = 5$ and $b = 40$.

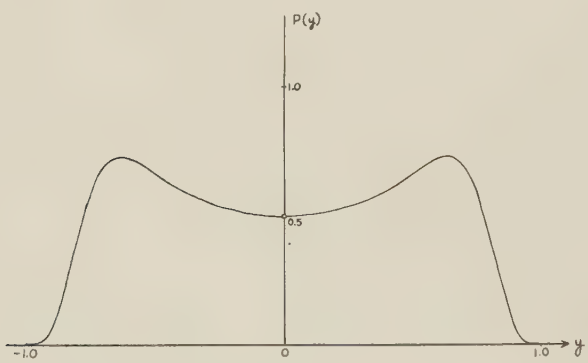


Fig. 2—Probability density of the output of an RC low-pass filter when the input is a binary random process, the axis-crossing intervals of which are statistically independent and which obey the probability density law shown in Fig. 1.

RC High-Pass Filter

Any solution for the *RC* low-pass filter may be adapted to the case of an *RC* high-pass filter, as shown by McFadden.[18]

Conclusion

This paper has described a new method for studying the output of an *RC* filter when the input is a stationary binary random process. Much work remains to be done concerning the properties of the solution, particularly the asymptotic approach to a Gaussian density. It is also conceivable that the method may be extended to other types of linear systems.

[14] McFadden, *op. cit.*, footnote 2, (20).
[15] McFadden, *op. cit.*, footnote 5, p. 22.
[16] J. A. McFadden, "The axis-crossing intervals of random functions," IRE Trans. on Information Theory, vol. IT-2, pp. 146–150 (5); December, 1956.
[17] W. M. Wonham, private communication.
[18] McFadden, *op. cit.*, footnote 2, (26).

Recurrent Events in a Bernoulli Sequence*

M. B. MARCUS†

Summary—The "point of regeneration" method is used to obtain simple sequential equations for determining the complete probability density function for multiple occurrences of events in a Bernoulli sequence. Both the independent and overlapping classes of recurrent events are included in the general framework of these equations. The equations also lead to the generating function for the probability distribution. This is used to obtain the expected recurrence times for the different classes of recurrent events.

A distinction is made between the probability distributions for the occurrence of the kth event at the nth trial and the occurrence of k events in n trials. The latter case is of primary concern.

The methods employed and the results obtained have extensive applications in problems in automatic control, communications, and information processing.

Introduction

THE techniques of functional equations, in particular, the "point of regeneration" method,[1,2] will be used to formulate equations for computing the probability density function of recurrent events in a Bernoulli process. An event is defined as any preestablished and distinguishable subsequence of the process. Recurrent events are events that repeat as the process continues. They are separated into two classes, independent and overlapping. We will find the probability that an event will occur k times in n trials of the process for all k and n and for each of these classes.

Feller[3] presents the generating functions for the probability density of the first occurrence of an event at the nth trial. Products of these generating functions, say k of them, give the generating function of the kth occurrence of the event at the nth trial. The kth occurrence of an event at the nth trial and k occurrences of the event in n trials are different. The latter case, which was suggested by counting successes in an experiment of length n, is the concern of this paper.

The generating function for the probability of k occurrences of an event in n trials will be found for both independent- and overlapping-type events. It will be used to derive generating functions for the occurrence of an independent event, as given by Feller, and for the occurrence of an overlapping event, a result previously unknown to the author. These two functions give the expected value for the occurrence of each type of event. A comparison of the expected values shows the relative rate at which these events accumulate in a Bernoulli process.

The significance of this paper is that it describes a simple and direct method for computing probability density functions for many important problems in communications. The method treats both independent and overlapping recurrent events and is not complicated by increasing the number of elements contained in an event.

The analysis considers both beginning the observations of a sequence from its start and after it has been in operation. A discussion of the physical importance of this analysis will clarify the meaning of all of these distinctions. The methods will be described first in the solution of two particular problems and then in general.

Recurrent Events of a Bernoulli Sequence

To illustrate recurrent events, consider a Bernoulli sequence of 1's and 0's. Each trial of the sequence has a probability p of turning up 1 and a probability $q = 1 - p$ of being 0. The trials are independent.

An event is any collection of 1's and 0's, such that at any trial we can tell whether or not the event has occurred. An example of an event might be "three consecutive 1's"; the trial at which the third consecutive 1 is obtained is the trial at which the event is said to occur. As the trials continue the event will, in general, occur again. The purpose of this paper is to derive a method of computing the probability that in n trials the event will recur k times, for all k and n.

Recurrent events can be viewed in two ways, as independent events or as overlapping events. If a recurrent event is *independent*, then after each occurrence of the event, the process is considered to begin newly again.

For *overlapping* events, each trial is viewed to see if the event has occurred, regardless of whether or not it occurred on the previous trials. If an event is defined as "three consecutive 1's," then four consecutive 1's represent one event and one 1 if the events are viewed independently, but two events if they are viewed as overlapping events. Both of these cases have important physical analogs.

In deriving the probabilities for k occurrences in n trials, we can either assume that the process begins at $n = 1$ or that the process is already underway and that at $n = 1$, a set of conditional probabilities already exists for the occurrence of the event.

Physical Examples

In each case of recurrent events, independent or overlapping, there are important physical examples. As an example of independent recurrent events, we might

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[1] T. E. Harris, "Some mathematical models for branching processes," *Proc. Second Berkeley Symp. on Math. Statistics and Probability*, pp. 305–328; 1951.

[2] R. Bellman and T. E. Harris, "On age dependent binary branching processes," *Ann. Math.*, vol. 55, pp. 280–295; March, 1952.

[3] W. Feller, "Probability Theory and its Applications," John Wiley and Sons, Inc., New York, N. Y., vol. 1; 1950.

consider memory erasing processes. These could be a decision followed by the clearance of a shift register or a servomechanism that changes into an unfavorable mode of operation when it receives a certain signal and which must be turned off and begun again fresh.

Overlapping recurrent events can characterize a system that needs a certain signal to begin operating and which will continue to operate as long as the signal is present. An obvious example is meteor burst communications.[4,5] Another example, one which led to this paper, was the analysis of a radar scheme[6] in which each input pulse was delayed and compared to the following pulse and was recorded only if they both exceeded a threshold. This is a case of overlapping recurrent events of two signals that exceed the threshold.

Event of Two Consecutive Occurrences

We will find the probability density function for an event consisting of two consecutive 1's for both the independent and the overlapping case. For the sake of simplicity, first consider that the process begins at $n = 1$.

We will define the following functions:

$R(n, k)$ = the conditional probability that in the next n trials the event will occur k times, given that the process begins at trial $n = 1$.

$P_0(n, k)$ = the conditional probability that in the next n trials the event will occur k times, given that the previous trial resulted in a 0.

$P_1(n, k)$ = the conditional probability that in the next n trials the event will occur k times, given that the previous trial resulted in a 1.

Consider the function $R(n, k)$, the probability of k occurrences in n trials, starting from the first trial. If the outcome of the first trial is a 1, $R(n, k)$ will be equal to $P_1(n - 1, k)$, the conditional probability of k occurrences in $(n - 1)$ trials, given that the previous pulse was a 1. If however, the outcome of the first trial is a 0, $R(n, k)$ will be equal to $P_0(n - 1, k)$. Recall that a 1 occurs with probability p and a 0 with probability $q = 1 - p$. Therefore

$$R(n, k) = pP_1(n - 1, k) + qP_0(n - 1, k). \quad (1)$$

If the process is in the conditional state corresponding to the probability $P_0(n, k)$, then according to the outcome of the next trial, it will go to either the state corresponding to $P_0(n - 1, k)$ or the state corresponding to $P_1(n - 1, k)$, *i.e.*,

$$P_0(n, k) = pP_1(n - 1, k) + qP_0(n - 1, k). \quad (2)$$

Therefore

$$R(n, k) = P_0(n, k), \quad (3)$$

and since we have not yet distinguished between independent and overlapping events, (3) applies to both of these cases.

Now consider that the process is in the conditional state that corresponds to the probability $P_1(n, k)$ and that independent events are being recorded. If the next trial results in a 1, then an event occurs. Since independent events are being recorded, the process must begin again at the next stage; *i.e.*, the process goes into the conditional state corresponding to the probability $R(n - 1, k - 1)$. By (3), this is $P_0(n - 1, k - 1)$. If, however, the next trial results in a 0, the event does not take place and the process goes into the state corresponding to $P_0(n - 1, k)$. Thus, in the case of independent recurrent events,

$$P_1(n, k) = pP_0(n - 1, k - 1) + qP_0(n - 1, k). \quad (4)$$

For the case of overlapping events, if the process is in $P_1(n, k)$ and the next trial results in a 1, then an event occurs and the process remains in the conditional state where the previous pulse is a 1 [(*i.e.*, if the trial results in a 1, $P_1(n, k)$ goes into $P_1(n - 1, k - 1)$]. Therefore, in the case of overlapping recurrent events,

$$P_1(n, k) = pP_1(n - 1, k - 1) + qP_0(n - 1, k). \quad (5)$$

The results of each of these cases will be summarized below.

Independent Recurrent Events of Two Consecutive 1's

Recall that $R(n, k)$ is the probability that in n trials, there will occur k events, and that $R(n, k) = P_0(n, k)$. In the case of independent recurrent events, we obtain the complete distribution of $R(n, k)$ from (2) and (4), *i.e.*,

$$\begin{aligned} P_0(n, k) &= pP_1(n - 1, k) + qP_0(n - 1, k) \\ P_1(n, k) &= pP_0(n - 1, k - 1) + qP_0(n - 1, k). \end{aligned} \quad (6)$$

This can be expressed as a single equation for $R(n, k)$,

$$\begin{aligned} R(n, k) &= P_0(n, k) \\ &= q\{P_0(n - 1, k) + pP_0(n - 2, k)\} \\ &\quad + p^2P_0(n - 2, k - 1). \end{aligned} \quad (7)$$

The initial values of $P_0(n, k)$ are easily obtained;

$$\begin{aligned} P_0(1, 0) &= 1 & P_0(1, 1) &= 0 \\ P_0(2, 0) &= 1 - p^2 & P_0(2, 1) &= p^2. \end{aligned} \quad (8)$$

Furthermore, $P_0(i, j) = 0$, $i < j$, $P_0(0, 0)$ is defined equal to 1. The process is undefined for $j < 0$. These values and (7) enable $R(n, k)$ to be solved recursively for all n and k.

[4] P. A. Forsyth, E. L. Vogan, D. R. Hansen, and C. O. Hines, "The principles of JANET—A meteor-burst communications system," Proc. IRE, vol. 45, pp. 1642–1657; December, 1957.

[5] L. L. Campbell, "Storage capacity in burst-type communication systems," Proc. IRE, vol. 45, pp. 1661–1666; December, 1957.

[6] M. B. Marcus, "Analysis of a Method to Permit the Operation of a Pulse Search Detection Radar in an Environment of Interfering Radar Units," The RAND Corp., Santa Monica, Calif., Res. Memo. RM-2375; May, 1959.

Overlapping Recurrent Events of Two Consecutive 1's

Similar to the previous case, $R(n, k) = P_0(n, k)$. The equations that govern the case of overlapping events are (2) and (5), *i.e.*,

$$\begin{aligned} P_0(n, k) &= pP_1(n-1, k) + qP_0(n-1, k) \\ P_1(n, k) &= pP_1(n-1, k-1) + qP_0(n-1, k). \end{aligned} \tag{9}$$

As an equation for $R(n, k)$ in terms of $P_0(n, k)$ alone, this is

$$\begin{aligned} R(n, k) &= P_0(n, k) \\ &= q\{P_0(n-1, k) + pP_0(n-2, k)\} \\ &+ p\{P_0(n-1, k-1) - qP_0(n-2, k-1)\}. \end{aligned} \tag{10}$$

The initial conditions are

$$\begin{aligned} P_0(0, 0) = 1 \quad P_0(1, 0) = 1 \quad P_0(2, 0) = 1 - p^2 \\ P_0(1, 1) = 0 \quad P_0(2, 1) = p^2, \end{aligned} \tag{11}$$

and $P_0(i, j) = 0$ when $i < j$. The process is undefined for $j < 0$.

Therefore, in this case of overlapping events, $R(n, k)$ can still be calculated from a single recursive equation.

In the next section, the problem of recurrent events will be expressed in generality including the two cases mentioned here.

Generalizations of Recurrent Events

Consider an event that consists of j elements, in which each element can either be a 1 or a 0. The sequence can be in any of the states r, $0 \leq r \leq j$, where the state r designates that the last r members of the sequence comprise the first r elements of the event. In this manner, the state 0 designates that the last member of the sequence contains no elements of the event.

Associated with each state r is a probability $P_r(n, k)$; $P_r(n, k)$ is the conditional probability that the sequence will contain k occurrences of the event in the next n trials, given that it is currently in state r.

In order for this problem to be deterministic, it must be stipulated that the sequence can be in one and only one state r at a given time and that given the history of the sequence, it is always possible to tell which state this is. Since the event is clearly established, the transition probabilities (p or q) of going from state r to state $r + 1$ are known. It is also assumed that if the process does not go on to state $r + 1$ it returns to state 0, unless of course the state $r + 1$ is the terminal state of the event. This case will be explained later. The stipulation that the process goes from state r to state 0 if it does not proceed in the pattern of the event is not necessary mathematically, but it is compatible with physical applications and generally simplifies the theory.

We assume that at the occurrence of an event of j elements, the process goes to the ith state. For independent events, $i = 0$. It is seen that the solution for the process is equally simple for any $0 \leq i \leq j - 1$. Notice that this extends the concept of overlapping events.

In order to simplify the expression of transition probabilities, we will consider that the event is j consecutive 1's. Since the 1's and 0's are independent, no loss of generality is imposed. We define:

$P_r(n, k)$ = the conditional probability that in the next n trials the event will occur k times, given that the previous r trials resulted in 1's.

When the event occurs, the process goes into state i. If a 0 occurs, the process returns to state 0. The equations describing this process are

$$\begin{aligned} P_0(n, k) &= pP_1(n-1, k) + qP_0(n-1, k) \\ P_1(n, k) &= pP_2(n-1, k) + qP_0(n-1, k) \\ &\vdots \\ P_i(n, k) &= pP_{i+1}(n-1, k) + qP_0(n-1, k) \\ &\vdots \\ P_{j-1}(n, k) &= pP_i(n-1, k-1) + qP_0(n-1, k). \end{aligned}$$

These can be reduced to one equation in $P_0(n, k)$, *i.e.*,

$$\begin{aligned} P_0(n, k) = q \sum_{r=1}^{j} p^{r-1} P_0(n-r, k) \\ + p^{j-i}\Big\{P_0(n-(j-i), k-1) \\ - q \sum_{r=1}^{i} P_0(n-(j-i), k-1)\Big\}. \end{aligned} \tag{13}$$

We have already established that $R(n, k) = P_0(n, k)$. The number of terms in the equation for $P_0(n, k)$ do not influence computing time. Thus, even in this general case, the complete probability density function can be obtained recursively from a single equation in P_0.

Generating Functions for $P_0(n, k)$

A generating function for $P_0(n, k)$ will be found; an event is j consecutive ones and the process returns to the state i after the event occurs. The generating function is defined as

$$U(s, t) = \sum_{n=j}^{\infty} \sum_{k=1}^{\infty} P_0(n, k) t^n s^k. \tag{14}$$

The counting for n begins at j because $n = j$ is the first trial at which the event can occur.

The equation for $U(s, t)$ is obtained from (13), with minor skills but much labor.

$$\begin{aligned} U(s, t) = \sum_{n=j}^{\infty} P_0(n, 1) t^n s \\ \cdot \left\{ \frac{1}{1 - \dfrac{p^{j-i} t^{j-i}(1 - t + qp^i t^{i+1})}{1 - t + qp^j t^{j+1}} s} \right\}. \end{aligned} \tag{15}$$

This can be rewritten

$$U(s, t) = \sum_{n=j} P_0(n, 1)t^n s \cdot \sum_{k=1}^{\infty} \left\{ \frac{p^{j-i}t^{j-i}(1 - t + qp^i t^{i+1})}{1 - t + qp^j t^{j+1}} \right\}^{k-1} s^{k-1}. \quad (16)$$

The generating function, as it is given in (16), is used to obtain important properties of recurrent events and to aid in calculating the values of $P_0(n, 1)$. We begin by considering the terms $P_0(n, k)$ in detail.

$P_0(n, 1)$ is the probability that the event occurs once in the first n trials. The occurrence of one event in the first n trials can happen in any of the following ways: the event occurs for the first time at the rth trial and does not occur again in the next $n - r$ trials, the event occurs first at the $r + 1$st trial and does not occur again in the next $n - (r + 1)$ trials, and so on for all r from 1 to n. Thus, the probability of one occurrence of the event *in* the first n trials is the convolution of the probability of the first occurrence of the event *at* the rth trial and the probability of no occurrences of the event *in* $n - r$ trials.

We must account for independent and overlapping events. At the start of a process, there is no history; overlapping is impossible for the first event. After the occurrence of the first event, the process returns to the state i. If $i = 0$, the events are independent. Notice that for the first occurrence of the event, both overlapping and independent events behave as independent events.

$P_0(n, 1)$ is equal to the convolution of the probability of the first occurrence of an independent event at the rth trial and the conditional probability that no event will occur in the next $n - r$ trials, given that an event has just occurred and the process is in state i.

Let $F_0(n)$ be the probability of the first occurrence of an independent event at the nth trial and let $W_0(t)$ be its generating function for all n.

Let $P_0(n, 0)$ be the conditional probability that no event will occur in n trials, given that an event has just occurred and the process returned to state i. Let $V_i(t)$ be the generating function for $\{P_0(n, 0)\}$.

Then

$$\sum_{n=j}^{\infty} P_0(n, 1)t^n = W_0(t)V_i(t). \quad (17)$$

$P_0(n, 2)$ is the probability of two occurrences of the event in the first n trials. The event will occur twice in the first n trials if the event occurs for the first time at the rth trial, the second time at the $r + l$th trial and does not occur from the $r + l + 1$st trial to the nth trial ($1 \leq r, l \leq n - 1, r + l \leq n$). Therefore, $P_0(n, 2)$ is the convolution of the probability of the first occurrence of an independent event at the rth trial, the conditional probability that the event will occur next at the lth trial, given that the event has just occurred and the process returned to state i, and the conditional probability that no event occurs in $n - (r + l)$ trials, given that an event has just occurred and the process returned to state i.

Let $F_i(n)$ denote the conditional probability that the event will occur next at the nth stage, given that the event has just occurred and the process returned to state i. Let $W_i(t)$ be the generating function for $\{F_i(n)\}$. Notice that for independent events $i = 0$, both $F_0(n)$ and $W_0(t)$ are identical to the expressions for the first occurrence of an independent event, as we would expect.

It follows that

$$\sum_{n=2j}^{\infty} P_0(n, 2)t^n = W_0(t)W_i(t)V_i(t), \quad (18)$$

substituting (17)

$$\sum_{n=2j}^{\infty} P_0(n, 2)t^n = \sum_{n=j}^{\infty} P_0(n, 1)t^n W_i(t). \quad (19)$$

Furthermore,

$$\sum_{n=kj}^{\infty} P_0(n, k)t^n = \sum_{n=j}^{\infty} P_0(n, 1)t^n W_i(t)^{k-1}. \quad (20)$$

A comparison of (16) and (20) shows that

$$W_i(t) = \frac{p^{j-i}t^{j-i}(1 - t + qp^i t^{i+1})}{1 - t + qp^j t^{j+1}}. \quad (21)$$

It follows directly, with $i = 0$, that

$$W_0(t) = \frac{p^j t^j(1 - pt)}{1 - t + qp^j t^{j+1}}. \quad (22)$$

Finally, because a return to state $j - 1$ is of particular interest, we display

$$W_{j-1}(t) = \frac{pt(1 - t + qp^{j-1}t^j)}{1 - t + qp^j t^{j+1}}. \quad (23)$$

The function $V_i(t)$ is the generating function for the probabilities that the first $j - i$ elements of the process are not all 1's and that no run of j consecutive 1's occurs by the nth trial of the process. It is obtained from recursion relations similar to (13).

$$V_i(t) = \frac{1 - p^{j-i}t^{j-i}}{1 - t + qp^j t^{j+1}}. \quad (24)$$

Thus, we can obtain the complete generating function for $U(s, t)$, where the process returns to state i after an event has occurred, $0 \leq i \leq j - 1$;

$$U(s, t) = \frac{p^j t^j(1 - pt)(1 - p^{j-i}t^{j-i})}{1 - t + qp^j t^{j+1}} s \cdot \sum_{k=1}^{\infty} \left(\frac{p^{j-i}t^{j-i}(1 - t + q^i t^{i+1})}{1 - t + qp^j t^{j+1}} \right)^{k-1} s^{k-1}. \quad (25)$$

Expected Number of Trials Required for the Occurrence of an Event

Expected values are meaningless when considering the occurrence of k events in n trials. However, we can find the expected number of trials required for the kth event to occur.

The generating function for the kth occurrence of an event at the nth trial is $W_0(t)\ W_i(t)^{k-1}$; call this $Z(t)$. Then

$d/dt\ Z(t)_{i=1}$ = expected number of trials required for k events to take place, when the process returns to state i after the occurrence of each event. (26)

Differentiating $Z(t)$ with respect to t and substituting $= 1$, we obtain

$$\frac{1 - p^i}{qp^i} + (k - 1)\frac{1 - p^{i-i}}{qp^i}. \tag{27}$$

Finally, we will examine the occurrence of events in the wo extremes of the process, that is, when the events are ndependent, $(i = 0)$, and when there is complete over- pping, $(i = j - 1)$. The expected number of trials equired for k occurrences are, respectively,

$$k \cdot \frac{1 - p^j}{qp^j} \tag{28}$$

nd

$$\frac{1 - p^j}{qp^j} + (k - 1)\cdot\frac{1}{p^j}. \tag{29}$$

If we consider that j (the number of elements in an vent) is large enough to make p^j negligible with respect o 1, and that k itself is large, then these two equations re approximated by

$$\frac{1}{q}\cdot\frac{k}{p^j} \tag{30}$$

nd

$$\frac{k}{p^j}. \tag{31}$$

Thus, in the extreme cases, overlapping events occur, n the average, $1/q$ times more frequently than independ- nt events.

Beginning Observations After a Process has Started

In the previous sections, we assumed that the process egan at $n = 1$. Now we will consider that the process as been in operation for a sufficient number of trials so hat even at the first observation, the event might occur. his implies that a portion of the sequence is stored in a emory and each observation samples the memory to see if the event has just occurred. Regardless of when we begin the observations, they continue, one for each stage of the process. Consistent with this extension, we define

$\pi(n, k)$ = the conditional probability that in the next n trials the event occurs k times, given that the process has been in operation a sufficient number of times so that it is possible for the event to occur at the first trial that is observed.

The value of $\pi(n, k)$ can be obtained by summing the conditional probabilities of making k events in n trials from each of the possible states that the process could be in, multiplied by the probability that the process is in that state. Thus,

$$\begin{aligned}\pi(n, k) &= qP_0(n, k) + (p - p^2)P_1(n, k) + \cdots \\ &\qquad + (p^{j-1} - p^j)P_{j-1}(n, k) \\ &= q\{P_0(n, k) + pP_1(n, k) + \cdots \\ &\qquad + p^{j-1}P_{j-1}(n, k)\};\end{aligned} \tag{32}$$

$\pi(n, k)$ can also be written in terms of P_0 alone.

It should be apparent that all the "states of beginning" between $R(n, k)$ and $\pi(n, k)$ can also be developed.

Conclusions

Functional equation techniques are easily applied to the problem of recurrent events, and simple computational equations are derived for obtaining the probability density function of the number of recurrences of the events. The techniques lead directly to the generating functions of the events, which have many more uses besides obtaining the rates at which the different processes develop.

These results are of value to workers in the fields of automatic control, information theory, and radar. Of greater significance, however, are the methods employed to obtain these results. They allow an increased breadth of approach to all sequential problems.

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Direction of Change with Refinement for Unweighted and Weighted Information-Entropy Functionals*

The information theory literature[1–3] contains numerous statements as to the sign to be prefixed before $\sum_i p_i \log p_i$ and $\sum_i p_i \log p_i/q_i$ in order to get information or entropy. In this note, we find the direction of change of generalized functionals with refinement of partitions of the underlying space. We suggest that the functionals be called information or entropy functionals according as they increase or decrease with refinement, and that the signs be chosen accordingly. The unweighted and weighted functionals behave oppositely in this respect.

Let Ω be a space and F a countably additive class of subsets of Ω. Let μ_0 be a measure, that is, a non-negative countably additive set function defined on F, and let $K\mu_0(A) = k \log \mu_0(A)$ for each $A \in F$, where k is a constant scale factor. If $B \in F$, $0 < \mu_0(B) < \infty$, and π is a countable disjoint partition of B into F-sets, we define the unweighted or *a posteriori*[4] functional $L\mu_0(\pi) = k/\mu_0(B) \sum_{A \in \pi} \mu_0(A) \log \mu_0(A)$, as the expectation of K_{μ_0} with respect to μ_0 over the partition π. Note that $\mu_0(A) < \infty$ for each $A \in \pi$ and that $\mu_0(A) = 0$ contributes nothing to the sum above.

To define functionals of the weighted or *a priori*[4] type, we consider a non-negative finitely superadditive set function μ in addition to the measure μ_0. That is, $\mu(A) \geqq 0$ for each $A \in F$ and $\mu(A_1 \cup A_2) \geqq \mu(A_1) + \mu(A_2)$ for $A_1 \cap A_2 = 0$, $A_1, A_2 \in F$. Then we form

$$K_{\mu_0,\mu}(A) = k \log \frac{\mu_0(A)}{\mu(A)}$$

and the expectation

$$L_{\mu_0,\mu}(\pi) = \frac{k}{\mu_0(B)} \sum_{A \in \pi} \mu_0(A) \log \frac{\mu_0(A)}{\mu(A)}$$

of $K_{\mu_0,\mu}$ with respect to μ_0, where as before π is a countable disjoint partition of B into F sets.

Let $\mu(A) = \Pi_{\nu \in M} \nu(A)$ be any non-empty product of measures $\nu \in M$. If M has two or more members, then μ is only superadditive, for

$$\mu(A_1 \cup A_2) = \prod_{\nu \in M} \nu(A_1 \cup A_2)$$

$$= \prod_{\nu \in M} [\nu(A_1) + \nu(A_2)]$$

$$\geqq \prod_{\nu \in M} \nu(A_1) + \prod_{\nu \in M} \nu(A_2).$$

If $\Omega = \Omega_1 \times \Omega_2$,

$$\mu_1(A_1) = \mu_0(A_1 \times \Omega_2),$$

$$\mu_2(A_2) = \mu_0(\Omega_1 \times A_2),$$

and μ is the product measure $\mu_1 \times \mu_2$, then $L_{\mu_0,\mu}(\pi)$ tends to the Kolmogorov information[5] as π is increasingly refined.

There are two essentially different situations possible for a measure μ_0 with countable disjoint partitions. In the discrete (atomic) case, there is a finest partition $\pi_{\max}$ such that $0 < \mu_0(B) < \infty$ for each $B \in \pi_{\max}$. In the continuous (atomless) case, there is no such partition. (A partition π' is said to be a refinement of π, in case each $B \in \pi$ is a countable union of sets $B' \in \pi'$).

We now derive some elementary inequalities. Let

$$\Phi(a_1, \cdots, a_n) = \left(\sum_{i=1}^n a_i\right) \log \left(\sum_{i=1}^n a_i\right) - \sum_{i=1}^n (a_i \log a_i).$$

Then

$$\Phi(a_1, \cdots, a_n, a_{n+1}) \geqq \Phi(a_1, \cdots, a_n) \geqq 0 \quad \text{for} \quad a_1, \cdots, a_{n+1} \geqq 0. \tag{1}$$

To prove this, note[6] that

$$\Phi(a_1, a_2) = \frac{1}{2\pi} \int_0^\infty \log\left(1 + \frac{a_1^2}{x^2}\right) \cdot \log\left(1 + \frac{a_2^2}{x^2}\right) dx \geqq 0.$$

By induction, it is easy to show that

$$\Phi(a_1, \cdots a_{n+1}) - \Phi(a_1, \cdots, a_n) = \Phi\left(\sum_{i=1}^n a_i, a_{n+1}\right) \quad \text{for} \quad n \geqq 2.$$

If $\{a_k\}$ is a non-negative sequence, we define

$$\Phi(a_1, a_2, \cdots)$$

as

$$\lim_{n \to \infty} \Phi(a_1, a_2, \cdots, a_n),$$

which exists and is non-negative by (1). This suffices for the unweighted case.

The more complicated weighted case is attacked in two stages. First, let

$$\Psi(a_1, \cdots, a_n; \quad b_1 \cdots b_n) = \left(\sum_{i=1}^n a_i\right) \log \left[\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\right] - \sum_{i=1}^n \left(a_i \log \frac{a_i}{b_i}\right),$$

for

$$a_1, \cdots, a_n \geqq 0, \quad b_1, \cdots, b_n > 0.$$

By induction, we can show that

$$\Psi(a_1, \cdots a_{n+1}; \quad b_1, \cdots b_{n+1}) - \Psi(a_1, \cdots a_n; \quad b_1, \cdots b_n) = \Psi\left(\sum_{i=1}^n a_i, a_{n+1}; \sum_{i=1}^n b_i, b_{n+1}\right)$$

for $n \geqq 2$. By a semi-geometric argument, we can show that $\Psi(a_1, a_2; b_1, b_2) \leqq 0$. Clearly the surface $z = \Phi(x, y)$ is a convex cone with vertex at the origin. Any tangent plane touches the surface in a ray whose xy trace is $b_1 x - b_2 y = 0$ for some $b_1, b_2 > 0$, and conversely. The corresponding tangent plane is

$$z = x \log\left(\frac{b_1 + b_2}{b_1}\right) + y \log\left(\frac{b_1 + b_2}{b_2}\right).$$

Since the surface is convex, every tangent plane is an upper bounding plane, so that

$$\Phi(a_1, a_2) - a_1 \log\left(\frac{b_1 + b_2}{b_1}\right) - a_2 \log\left(\frac{b_1 + b_2}{b_2}\right) \leqq 0.$$

* Received by the PGIT, December 8, 1958.

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[5] A. N. Kolmogorov, "On the Shannon theory of information transmission in the case of continuous signals," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, p. 103; December, 1956.

[6] D. Bierens de Haan, "Nouvelle Table des Integrales Definies," Stechert, New York, N. Y., table 33, no. 5, p. 59; 1939.

Clearly then

$$\Psi(a_1, \cdots, a_{n+1}; b_1, \cdots, b_{n+1}) \leqq$$

$$\Psi(a_1, \cdots, a_n; b_1, \cdots, b_n) \leqq 0. \quad (2)$$

Hence $\Psi(a_1, a_2, \cdots; b_1, b_2, \cdots) = \lim_{n\to\infty} \Psi(a_1, \cdots, a_n; b_1, \cdots, b_n)$ exists and is $\leqq 0$ for sequences $\{a_k\}$, $\{b_k\}$, $a_k \geqq 0$, $b_k > 0$.

Second, let

$$\Psi(a_1, \cdots a_n; \quad b_1, \cdots b_n; \quad c_n) =$$

$$\left(\sum_{i=1}^{n} a_i\right) \log \left[\frac{\sum_{i=1}^{n} a_i}{c_n}\right] - \sum_{i=1}^{n} a_i \log \frac{a_i}{b_i}$$

and note that

$$\Psi(a_1, \cdots a_{n+1}; b_1, \cdots b_{n+1}; c_{n+1})$$

$$= \Psi(a_1, \cdots a_n; b_1, \cdots b_n; c_n)$$

$$+ \Psi\left(\sum_{i=1}^{n} a_i, c_n; a_{n+1}, b_{n+1}\right)$$

$$+ \left(\sum_{i=1}^{n+1} a_i\right) \log \left(\frac{c_n + b_{n+1}}{c_{n+1}}\right).$$

Thus $c_{n+1} \geqq c_n + b_{n+1}$ implies

$$\Psi(a_1, \cdots a_{n+1}; b_1, \cdots b_{n+1}; c_{n+1})$$

$$\leqq \Psi(a_1, \cdots a_n; b_1, \cdots b_n; c_n). \quad (3)$$

Since

$$\Psi(a_1, a_2; b_1, b_2; c)$$

$$= \Psi(a_1, a_2; b_1, b_2)$$

$$+ (a_1 + a_2) \log \left(\frac{b_1 + b_2}{c}\right),$$

we have $\Psi(a_1, a_2, b_1, b_2, c) \leqq 0$ whenever $c \geqq b_1 + b_2$. Hence if $c_2 \geqq b_1 + b_2$, $c_{n+1} \geqq c_n + b_{n+1}$ for each $n \geqq 2$, then

$$\Psi(a_1, a_2, \cdots;$$

$$b_1, b_2, \cdots; c_1, c_2, \cdots)$$

$$= \lim_{n\to\infty} \Psi(a_1, \cdots, a_n; b_1, \cdots b_n; c_n)$$

exists and is $\leqq 0$.

The elementary inequalities yield inequalities for the unweighted and weighted functionals as partitions are refined.

Theorem 1: Let π' be a refinement of π. If

$$k < 0,$$

then

$$L_{\mu_0}(\pi') \geqq L_{\mu_0}(\pi)$$

and if

$$k > 0, L_{\mu_0}(\pi') \leqq L_{\mu_0}(\pi).$$

Proof: If $\pi = \{B_i\}$, $\pi' = \{C_j'\}$ are partitions of $A \in F$, then for each i there is a set N_i of indexes such that $B_i = U_{j\in N_i} C_j'$, by definition of refinement. From (1),

$$0 \leqq \sum_{j\in N_i} \mu_0(C_j') \log \sum_{j\in N_i} \mu_0(C_j')$$

$$- \sum_{j\in N_i} \mu_0(C_j') \log \mu_0(C_j')$$

$$= \mu_0(B_i) \log \mu_0(B_i)$$

$$- \sum_{j\in N_i} \mu_0(C_j') \log \mu_0(C_j').$$

Summing over i yields

$$0 \leqq \sum_i \mu_0(B_i) \log \mu_0(B_i)$$

$$- \sum_j \mu_0(C_j') \log \mu_0(C_j')$$

so that for $k > 0$,

$$L_{\mu_0}(\pi') = \frac{k}{\mu_0(A)} \sum_j \mu_0(C_j') \log \mu_0(C_j')$$

$$\leqq \frac{k}{\mu_0(A)} \sum_i \mu_0(B_i) \log \mu_0(B_i)$$

$$= L_{\mu_0}(\pi).$$

This is to be expected. However, intuition about the weighted functionals $L_{\mu_0,\mu}$ is weaker. The μ_0 dependence of $L_{\mu_0,\mu}$ would suggest the same behavior as that of L_{μ_0} and the μ dependence would suggest the opposite behavior.

We now prove that if the weighting set function μ is super-additive, then the weighted functional $L_{\mu_0,\mu}$ behaves oppositely to L_{μ_0} for a given choice of k.

Theorem 2: Let μ be a non-negative super-additive set function. If $k < 0$, then

$$L_{\mu_0,\mu}(\pi') \leqq L_{\mu_0,\mu}(\pi),$$

and if

$$k > 0, L_{\mu_0,\mu}(\pi') \geqq L_{\mu_0,\mu}(\pi).$$

Proof: Write $\{C_j' : j\in N_i\}$ as $\{D^i_j\}$. Let

$$a_i = \mu_0(D_i^{(i)}),$$

$$b_i = \mu(D_i^{(i)}), \qquad c_i = \mu\left(\bigcup_{k=1}^{i} D_k^{(i)}\right).$$

Note that

$$c_2 = \mu(D_1^{(i)} \cup D_2^{(i)}) \geqq \mu(D_1^{(i)})$$

$$+ \mu(D_2^{(i)}) = b_1 + b_2,$$

$$c_{n+1} = \mu\left(\bigcup_{k=1}^{n} D_k^{(i)} \cup D_{n+1}^{(i)}\right)$$

$$\geqq \mu\left(\bigcup_{k=1}^{n} D_k^{(i)}\right) + \mu(D_{n+1}^{(i)})$$

$$= c_n + b_{n+1}$$

since the partition $D_j^{(i)}$, of B_i is disjoint.

Moreover

$$\mu(B_i) = \mu(\bigcup_i D_i^{(i)}) \geqq \mu(\bigcup_{i>n} D_i^{(i)})$$

$$+ \mu\left(\bigcup_{i=1}^{n} D_i^{(i)}\right) \geqq \mu\left(\bigcup_{i=1}^{n} D_i^{(i)}\right),$$

so

$$\mu(B_i) \geqq \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} D_i^{(i)}\right).$$

Hence by (3) and the continuity of $x \log x/y$ for $x, y > 0$, we have

$$0 \geqq \Psi(a_1, a_2, \cdots ; b_1, b_2, \cdots ; c_1, c_2, \cdots)$$

$$= [\sum_j \mu_0(D_j^{(i)})] \cdot \log \frac{\left(\sum_j \mu_0(D_j^{(i)}) \right)}{\left(\lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} D_j^{(i)} \right) \right)}$$

$$- \sum_j \mu_0(D_j^{(i)}) \log \frac{\mu_0(D_j^{(i)})}{\mu(D_j^{(i)})}$$

$$\geqq \mu_0(B_i) \log \frac{\mu_0(B_i)}{\mu(B_i)} - \sum_j \mu_0(D_j^{(i)}) \log \frac{\mu_0(D_j^{(i)})}{\mu(D_j^{(i)})}$$

Summation over i and multiplication by $k/\mu_0(A)$ gives the stated result.

The above results can be generalized to conditional functionals. If, as we suggest, information is to be a nondecreasing functional of partition refinement, then any non-negative linear combination of L_{u_0}, $L_{u'_0}$, $\cdots$ (with negative constants) and $L_{u_0,u}$, $L_{u'_0,u'}$, $\cdots$ (with positive constants) for measures μ_0, μ'_0, $\cdots$ and superadditive set functions μ, μ', $\cdots$ would supply a suitable information functional. On the other hand, any non-negative linear combination of L_{u_0}, $L_{u_0'}$, $\cdots$ (with positive constants) and $L_{u_0,u}$, $L_{u_0',u'}$, $\cdots$ (with negative constants) would supply a suitable entropy functional.

We now make an elementary application of these remarks. Let ξ, η be discrete random variables, and let

$$P_{ij} = P(\xi = \xi_i, \eta = \eta_j),$$

$$H(\xi, \eta) = -\sum_{i,j} P_{ij} \log P_{ij}.$$

The functionals $H(\xi)$, $H(\eta)$, $H(\xi|\eta)$, $H(\eta|\xi)$, and $R(\xi, \eta) = H(\xi, \eta) - H(\xi) - H(\eta)$ are defined as usual. From the point of view expressed above, $H(\xi, \eta)$, $-H(\xi|\eta)$, $-H(\eta|\xi)$, $H(\xi)$, $H(\eta)$, and $-R(\xi, \eta)$ have the nature of information. Kolmogorov[5] calls $I(\xi, \eta) = -R(\xi, \eta)$ the quantity of information in ξ relative to η.

From the monotone character of L_{μ_0} and $L_{\mu_0,\mu}$, and the fact that disjoint countable (or disjoint finite) partitions of a set form a directed system[7] (net) with respect to ordering by refinement, it follows that limits with respect to partition refinement always exist and define set functions.

In the discrete (atomic) case, the maximal partition furnishes the limiting value.

In the continuous (atomless) case, the limits are nontrivial. Under suitable restrictions on μ_0, μ the author has shown that the limiting set functions $L_{\mu_0,\mu}$ possess the properties usually ascribed to information and entropy, and are finite. This generalizes some results of Kolmogorov[5] and his associates.

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[7] J. L. Kelley, "General Topology," D. Van Nostrand Co., Inc., New York, N. Y.; 1956.

Correction to "A Note on Angle Modulation by a Mixture of a Periodic Function and Noise"

P. R. Karr, author of this Correspondence item which appeared in the September, 1959, issue of these Transactions, has called the following to the attention of the Editor.

The sentence immediately after (13) should read: "The latter approximation is not inconsistent with the former."

In (3), ω_0 should be ω.

Above (6), the symbol $X \to \infty$ should be deleted.

Further Comments on "A Markoff Envelope Process"*

In an article by Pierce[1] it was shown that when a singly tuned RLC filter is excited with white Gaussian noise, the envelope of the resulting narrow-band noise constitutes a first-order Markoff process. A concise alternative proof of this same fact, based on use of the Fokker-Planck equation, was given by Helstrom in a recent article, and a third proof, essentially intuitive in character, was offered by Isley[2]. This last proof contains a number of apparent errors which are summarized in what follows. We will take "Markoff" to mean first-order Markoff in the remaining material.

In the third paragraph of Isley's letter,[2] it is stated that "Since the envelope detector is a zero memory device, the dimensionality of the process at its output cannot, by dint of definition, exceed the dimensionality of the process at its input." This statement is in error for two reasons: First, the implied definition is incorrect since it is necessary that the nonlinear device operating on the input be not only zero-memory but also one-to-one. Second, the envelope detector, considered as a physical device, is neither zero-memory nor one-to-one. An example is given in Appendix I showing that the square of a discrete Markoff process need not be Markoff. A similar proof, due to Price[3] and not reproduced here, shows that the square of the nonzero mean Markoff Gaussian process is not Markoff. These two examples are sufficient to show the necessity for a monotonic transformation of the input to preserve the Markoff character. That the actual envelope detector is not memoryless or monotonic is obvious from the fact that it must have a memory extending over at least a few cycles of the center frequency, and from the fact that it is insensitive to the actual phase of the input signal.

As was explicitly stated by Pierce,[4] the in-phase and quadrature components of the given narrow-band signal are themselves Markoff, and statistically identical. In the latter part of the third paragraph of Isley's letter it is suggested that it then follows trivially that the envelope is also Markoff. This is not true since examples can be constructed showing that the length of a two-dimensional Markoff vector (having statistically identical, independent orthogonal components) need not be Markoff. Such an example is given in Appendix II. Alternatively, one might base the demonstration on the fact that the square of the zero mean, Markoff Gaussian process is also Markoff;[3] however, it would then be necessary to show that the sum of two such squared components was also Markoff. This is not trivial either, since it is not true in general that sum processes are of the same order as the components. An illustration of this is again to be found in the example of Appendix II.

The interesting conjecture about higher order Markoff envelope processes made in the last paragraph of Isley's letter may quite possibly be true; it would certainly require a more rigorous proof than was presented in the letter.

It should be pointed out that there is a typographical omission in the original article.[5] Eq. (19) should have in the denominator of the right-hand side, the multiple product appearing on the right-hand side of (18).

* Received by the PGIT, August 9, 1959.

[1] J. N. Pierce, "A Markoff envelope process," IRE Trans. on Information Theory, vol. IT-4, pp. 163–166; December, 1958.

[2] C. W. Helstrom and C. T. Isley, "Two notes on a Markoff envelope process," IRE Trans. on Information Theory, vol. IT-5, pp. 139–140; September, 1959.

[3] R. Price, M.I.T. Lincoln Lab., Lexington, Mass., unpublished material.

[4] J. N. Pierce, *op. cit.*, p. 166.

[5] Pierce, *op. cit.*, p. 165, (19).

Appendix I

Non-Markoff Square of Markoff Process

Let $x_1, x_2, \cdots, x_m$, be a stationary discrete Markoff process, where x can assume the values -1, 1, and 3. Let the matrix of transition probabilities, $p(x_{m+1} \mid x_m)$, be given by

$$\begin{array}{c} \\ -1 \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} -1 & 1 & 3 \end{array} \\ \begin{bmatrix} 1-\alpha-\alpha^2 & \alpha & \alpha^2 \\ \alpha & 1-2\alpha & \alpha \\ \alpha^2 & \alpha & 1-\alpha-\alpha^2 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ x_m \\ \\ \\ \end{array}.$$

x_{m-1}

We find immediately that $p(-1) = p(1) = p(3) = 1/3$.

Let $y_m = x_m^2$, and let $p_x(x_1, x_2, \cdots, x_m)$ denote the probability of a particular sequence of x values, and let $p_x(x_{m+1} \mid x_1, x_2, \cdots, x_m)$ denote the conditional probability of a particular value of x_{m+1}.

Then, for example, we have

$$\begin{aligned} p_y(1, 1, 9) &= p_x(1, 1, 3) \\ &\quad + p_x(1, -1, 3) \\ &\quad + p_x(-1, 1, 3) \\ &\quad + p_x(-1, -1, 3) \\ &= \tfrac{1}{3}\alpha(1 - \alpha^3) \\ p_y(1, 1) &= p_x(1, 1) + p_x(1, -1) \\ &\quad + p_x(-1, 1) \\ &\quad + p_x(-1, -1) \\ &= \tfrac{1}{3}(2 + \alpha)(1 - \alpha) \\ p_y(1, 9) &= p_x(1, 3) \\ &\quad + p_x(-1, 3) \\ &= \tfrac{1}{3}\alpha(1 - \alpha) \\ p_y(1) &= p_x(1) + p_x(-1) \\ &= \tfrac{2}{3}. \end{aligned}$$

It follows, then, that

$$\begin{aligned} p_y(9 \mid 1, 1) &= \frac{p_y(1, 1, 9)}{p_y(1, 1)} \\ &= \frac{\alpha + \alpha^2 + \alpha^3}{2 + \alpha} \\ p_y(9 \mid 1) &= \frac{p_y(1, 9)}{p_y(1)} \\ &= \frac{\alpha + \alpha^2}{2}. \end{aligned}$$

Since $p_y(9 \mid 1, 1) \neq p_y(9 \mid 1)$, $\{y\}$ is not Markoff.

Appendix II

Non-Markoff Sum of Markoff Processes and Non-Markoff Length of Markoff Vector

Let $x_1, x_2, \cdots, x_m, \cdots$, and $y_1, y_2, \cdots, y_m, \cdots$, be two statistically independent, statistically identical stationary Markoff sequences, where x, y can assume the values 0, 1, and 2. Let the transition probability matrix be

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1-2\alpha & \alpha & \alpha \\ \alpha & 1-2\alpha & \alpha \\ \alpha & \alpha & 1-2\alpha \end{bmatrix} \end{array}.$$

The x (or y) values are then equiprobable.

Let $z_1, z_2, \cdots, z_m, \cdots$, be the sum sequence with $z_m = x_m + y_m$. Using the same notation as in Appendix I we find

$$\begin{aligned} p_z(0, 2, 0) &= p_x(0, 2, 0)p_y(0, 0, 0) \\ &\quad + p_x(0, 1, 0)p_y(0, 1, 0) \\ &\quad + p_x(0, 0, 0)p_y(0, 2, 0) \\ &= \frac{2\alpha^2 - 7\alpha^4 + 8\alpha^6}{9} \\ p_z(0, 2) &= p_z(2, 0) \\ &= p_x(0, 0)p_y(0, 2) \\ &\quad + p_x(0, 1)p_y(0, 1) \\ &\quad + p_x(0, 2)p_y(0, 0) \\ &= \frac{2\alpha - 3\alpha^2}{9} \\ p_z(2) &= \tfrac{1}{3}. \end{aligned}$$

We then have

$$\begin{aligned} p_z(0 \mid 0, 2) &= \frac{p_z(0, 2, 0)}{p_z(0, 2)} \\ &= \frac{2\alpha^2 - 7\alpha^4 + 8\alpha^6}{2\alpha - 3\alpha^2} \\ p_z(0 \mid 2) &= \frac{p_z(2, 0)}{p_z(2)} \\ &= \frac{2\alpha - 3\alpha^2}{3}. \end{aligned}$$

Since $p_z(0 \mid 0, 2) \neq p_z(0 \mid 2)$, the sum process is not Markoff.

Now let $u = \sqrt{x}$, $v = \sqrt{y}$, $w = \sqrt{z}$. If we identify u and v with the orthogonal components of a two dimensional vector, then w is the length of the vector. It then follows that the length need not be Markoff even though the vector is.

Acknowledgment

The author would like to acknowledge the help of Dr. R. Price of M.I.T. Lincoln Laboratory, who supplied virtually all of the material on Markoff processes cited here.

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*Reply by Mr. Isley**

The term "envelope detector" in the third paragraph of Isley[6] was intended to apply particularly to the equivalent detection process which yields the envelope function assumed by Pierce.[1]

It is recognized that, in terms of actual physical conditions, envelope detectors or perhaps more properly, envelope detection processes introduce at a minimum an increase of one dimension in the process even though the nonlinear element proper may have zero memory. It may be remarked that the analysis advanced by Pierce is accordingly only an approximation, since he has neglected the memory introduced by the post detection filter associated with the nonlinear element. It might be intuitively argued that this approximation is valid if it is postulated that a suitable postdetection filter is available: a filter having negligible effect on the so-called envelope components while providing an arbitrarily high suppression of all other components out of the nonlinear device. It may also be noted that the assumption regarding orthogonality of the "envelope" components also introduces an approximation if a rigorous mathematical interpretation of the assumed physical process is required.

Practical physical conditions notwithstanding, the process postulated by Pierce may be readily established as a mathematical model. In such a model, the "envelope detector" might be a three-port device having two inputs and one output. The orthogonal components of the input process are respectively applied to the inputs. These component processes, as previously noted, are independent, statistically identical, and two-dimensional

* Received by the PGIT, September 28, 1959.
[6] Helstrom and Isley, *op. cit.* (see the third paragraph in the second note.)

Gaussian. The detection process is as follows: each input component is squared; the resulting squares are linearly summed; the square root of the resulting sum is effected. For such a model, the detection process obviously introduces no memory.

The fact that samples of the envelope function derived from the components of the input process constitute a first-order Markoff process is readily demonstrated as follows.

It may be noted that the components of the input process $X(t)$ and $Y(t)$, respectively, may be expressed as:

$$X(t) = \int_0^t e^{-\alpha(t-\tau)} U(\tau)\, d\tau$$

$$Y(t) = \int_0^t e^{-\alpha(t-\tau)} V(\tau)\, d\tau$$

where $U(t)$ and $V(t)$ are independent, statistically identical, first-order Gaussian processes.

It follows that, for $t_N > t_{N-1}$,

$$X(t_N) = p(t_N, t_{N-1}) + b_{N,N-1} X(t_{N-1})$$

$$Y(t_N) = q(t_N, t_{N-1}) + b_{N,N-1} Y(t_{N-1})$$

where:

$$p(t_N, t_{N-1}) = \int_{t_{N-1}}^{t_N} e^{-\alpha(t_N-\tau)} U(\tau)\, d\tau$$

$$q(t_N, t_{N-1}) = \int_{t_{N-1}}^{t_N} e^{-\alpha(t_N-\tau)} V(\tau)\, d\tau$$

$$b_{N,N-1} = e^{-\alpha(t_N - t_{N-1})}$$

$$X(t_{N-1}) = \int_0^{t_{N-1}} e^{-\alpha(t_{N-1}-\tau)} U(\tau)\, d\tau$$

$$Y(t_{N-1}) = \int_0^{t_{N-1}} e^{-\alpha(t_{N-1}-\tau)} V(\tau)\, d\tau.$$

As a fact of fundamental significance in the conclusion to follow, it should be noted that the *memory* in both of the Gaussian processes defined respectively by the functions $p(t_N, t_{N-1})$ and $q(t_N, t_{N-1})$ is confined *solely* to the epoch $t_N \geq t \geq t_{N-1}$.

Noting that:

$X(t_N)$, $Y(t_N)$ are the rectilinear components of the two-dimensional vector $\overrightarrow{R_N}$,

$X(t_{N-1})$, $Y(t_{N-1})$ are the rectilinear components of the two-dimensional vector $\overrightarrow{R_{N-1}}$,

and defining

$p(t_N, t_{N-1})$, $q(t_N, t_{N-1})$ to be the rectilinear components of the two-dimensional vector $\overrightarrow{W_{N,N-1}}$,

the following vector relationship may be obviously established:

$$\overrightarrow{R_N} = b_{N,N-1} \overrightarrow{R_{N-1}} + \overrightarrow{W_{N,N-1}}.$$

It then obviously follows that the joint probability distribution

$$p(R_N, R_{N-1}, \theta_{N,N-1}; \tau_{N,N-1})$$

where

$$\theta_{N,N-1} = \theta_N - \theta_{N-1}$$

$$\tau_{N,N-1} = t_N - t_{N-1}$$

$$\overrightarrow{R_N} = R_N/\underline{\theta_N}$$

$$\overrightarrow{R_{N-1}} = R_{N-1}/\underline{\theta_{N-1}}$$

constitutes a complete set of statistics for the process $\{\overrightarrow{R_N}, \overrightarrow{R_{N-1}}\}$. The fact that the marginal distribution $p(R_N, R_{N-1}; \tau_N, {}_{N-1})$ establishes the process $\{R_N, R_{N-1}\}$ to be Markoff follows trivially.

The examples cited in Pierce's comments are interesting, but do not appear relevant to the processes considered in Pierce's paper and Isley's note.

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*Letter from Mr. Pierce**

Mr. Isley is to be complimented for suggesting a more direct form. I would like to offer an even shorter proof which I believe contains the substance of his.

Let the subscripts 1, 2, $\cdots$, N correspond to time instants $t_1 < t_2 < \cdots < t_N$. Let R and θ be the amplitude and phase, and let x and y be the inphase and quadrature components of the previously described narrowband process. Then

$$p(R_N \mid R_1, R_2, \cdots, R_{N-1})$$

$$= p(R_N \mid R_1, R_2, \cdots, R_{N-1}, \theta_{N-1}) \quad \text{for any } \theta_{N-1} \qquad (1)$$

$$= p(R_N \mid R_1, R_2, \cdots, x_{N-1}, y_{N-1}) \qquad (2)$$

$$= p(R_N \mid x_{N-1}, y_{N-1}) \qquad (3)$$

$$= p(R_N \mid R_{N-1}). \qquad (4)$$

(1) follows from the fact that the envelope distributions are independent of arbitrary fixed phase shifts of the process. Selecting any particular value for the phase angle at the immediately previous sample point fixes the value of the inphase and quadrature components at that sample point as indicated by (2). But since x and y are known to be statistically identical zero mean Markoff processes, all available information about x_N and y_N (and hence R_N) is given by knowledge of x_{N-1} and y_{N-1} as implied by (3). Finally, since the choice of θ_{N-1} was completely arbitrary, x_{N-1} and y_{N-1} are determined solely by R_{N-1} which gives the desired result (4), and completes the proof.

* Received by the PGIT, October 10, 1959.

This proof can *not* be extended to higher order Markoff processes. To do this would require an assumption of arbitrary phase angles at two or more previous sample points; this assumption is not permissible.

The same type of proof can be used to show in a simple fashion that the square of the zero mean Gaussian Markoff process is also Markoff:

Let x be the zero mean Gaussian Markoff process, let $v = x^2$, and let $s = |x|/x$. (s is plus or minus one corresponding to positive or negative x.) Then

$$p(v_N \mid v_1, v_2, \cdots, v_{N-1})$$

$$= p(v_N \mid v_1, v_2, \cdots, v_{N-1}, s_{N-1}) \qquad (5)$$

$$= p(v_N \mid v_1, v_2, \cdots, x_{N-1}) \qquad (6)$$

$$= p(v_N \mid x_{N-1}) \qquad (7)$$

$$= p(v_N \mid v_{N-1}). \qquad (8)$$

The supporting reasoning takes an identical form to that following (1)–(4).

It is also obvious that whether or not the phase angle is Markoff (for the narrow band process) this type of proof can not be used to demonstrate the fact. The assumption of an arbitrary envelope sample value is equivalent to assuming an arbitrary gain factor and consequently assuming a new, and unknown, mean power for the narrowband process.

J. N. Pierce

Error-Correcting Codes for an Asymmetric Nonbinary Channel*

In transmission of messages with pulse amplitude modulation through a noisy channel with no memory, detection and correction schemes of small errors [(± 1)-errors] for a symmetric nonbinary channel have been extensively studied in the literature.[1–4] However, it is observed that the intrinsic transmission characteristics in some physical devices as well as in some communication systems are of an asymmetric nature. For instance, one example is a magnetic tape storage unit in a large-scale computer;[5] another is the dissimilar fading on mark and space channels which

* Received by the PGIT, October 19, 1959. This work was supported by Natl. Science Foundation Grant G-3676.

[1] M. J. E. Golay, "Notes on digital coding," Proc. IRE, vol. 37, p. 657; June, 1949.

[2] W. Ulrich, "Non-binary error correcting codes," *Bell Sys. Tech. J.*, vol. 36, pp. 1341–1388; November, 1957.

[3] C. Y. Lee, "Some properties of nonbinary error-correcting codes," IRE Trans. on Information Theory, vol. IT-4, pp. 77–82; June, 1958.

[4] H. S. Shapiro and D. L. Slotnick, "On the mathematical theory of error-correcting codes," *IBM J. Res. and Dev.*, vol. 3, pp. 25–34; January, 1959.

[5] E. G. Newman and L. O. Nippe, "Simulation of an Information Channel on the IBM 704 Computer," IBM Tech. Publication, Tr 00.01000.677; January, 1959.

s a type of operation that is sometimes sought in scatter circuits.[6]

In the previous work,[7,8] a correcting code was developed to correct only one type of error, 1-error or 0-error, for an asymmetric binary channel. As an extension of the work, a correcting code is proposed to correct only one type of small-errors,[2] (+1)-errors or (−1)-errors, for a nonbinary channel which has a transmission characteristic such that the probability of (−1)-error is much greater than that of (+1)-error or vice versa. This assumption may easily be implemented in practice by setting the position of brushes for a disk code generator,[9] or by setting a bias or threshold level.

Let us take an example of a ternary channel shown in Fig. 1, where

α = probability of (−1)-error

and (mod 3)

β = probability of (+1)-error.

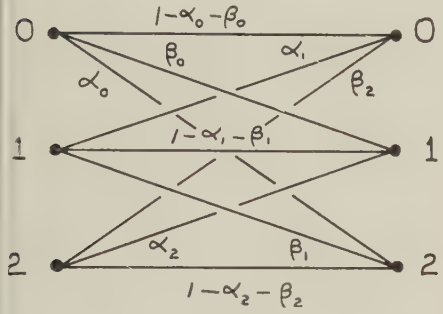


Fig. 1—A ternary channel.

Referring to Fig. 1, if $\alpha_i = \beta_j$ for all i and j, then the channel is said to be symmetric. Otherwise, it may be called an asymmetric channel. The type of channel considered in this paper is a *uniformly asymmetric channel* which has transmission characteristics as follows:

$$\alpha_i = \alpha_j$$
$$\beta_i = \beta_j \quad \text{for all } i \text{ and } j. \tag{1}$$
$$\alpha_i \gg \beta_j \quad (\text{or } \alpha_i \ll \beta_i)$$

Let k be a positive integer, with $k \geq 2$, and assume that each letter[3] (or each digit) of an n-length word takes k-values 0, 1, 2, $\cdots$, $k-1$. Then, it was shown by Lee[3] that the distance (a circular metric), between two words $s = (s_1, s_2, \cdots, s_n)$ and $t = (t_1, t_2, \cdots, t_n)$, required to correct e-tuple (± 1)-errors is

$$\rho(s, t) = \sum_{i=1}^{n} \rho(s_i, t_i) \geq 2e + 1 \tag{2}$$

where $\rho(s_i, t_i) = \text{Min.}\{s_i - t_i, t_i - s_i\}$ mod k. However, if a channel exhibits a highly asymmetric transmission characteristic, it may sometimes be sufficient to correct only one type of small error [(+1)-error *or* (−1)-error] and the distance requirement (2) may be considerably relaxed. The nature of the relaxation may be clearly seen in the case where a channel possesses a highly asymmetric uniform transmission characteristic.[7,8] The weight-distance requirements for a pair of words of an e-tuple (−1)-error [or (+1)-error] correcting code are found to be, for $n \geq 2$,

$$\Delta W(s, t) = |\, W(s) - W(t) \,| \geq e + 1 \tag{3a}$$

or

$$\rho(s, t) \geq 2(e + 1) - \Delta W(s, t) \tag{3b}$$

or

$$\rho(s_i, t_i) \geq e \quad \text{when} \quad l \geq e \geq 2 \tag{3c}$$

where $W(s) = \sum_{i=1}^{n} s_i$ mod k, and $l = k/2$ when k is even and $l = (k-1)/2$ when k is odd. The weight-distance requirement (3) for asymmetric small-error correction may readily be deduced by following the arguments used in the previous work[6,7] for $k = 2$.

Single (−1)-Error Correcting Codes

For single (−1)-error correction [or (+1)-error correction], (3) is rewritten as

$$\Delta W(s, t) \geq 2, \quad \text{or} \quad \rho(s, t) \geq 3. \tag{4}$$

Let us use the following notations:

$S_k(n)$: a set of all k^n sequences.

Prefix : the first r letters of a sequence of length n.

Suffix : the last $(n-r)$ letters of a sequence of length n.

$C_k(n, d)$: a code which is designed for a symmetric nonbinary channel where $d \geq 2e + 1$.

$\oplus$: position-by-position addition mod k.

Then, a set of code words may be generated by the following scheme:

$$[\underbrace{x_i}_{\text{prefix}} \quad \underbrace{x_i \oplus c_j}_{\text{suffix}}] \quad \text{for all } i \text{ and } j \tag{5}$$

where $x_i \in S_k(r)$, $c_j \in C_k(n, 3)$, $i = 1, 2, \cdots, k^r$, $j = 1, 2, \cdots, b_k(n, 3)$ or $a_k(n, 3)$,[10] and $r = n/2$ for n = even and $r = (n-1)/2$ for n = odd. When x_i and c_j are of different lengths, the left-most position is aligned before addition.

The proof that the generation scheme (5) satisfies the weight-distance requirement (4), is given in the Appendix. It is shown in the proof that the proposed scheme (5) satisfies the requirement (4) for all values of k, where $k \geq 2$. Several examples are given for $k = 3$ for illustration and a k-valued code for $k > 3$ will follow the same generation scheme.

Example 1: $n = 5$, and $k = 3$. Then, prefixes are all elements in $S_3(2)$ and suffixes are generated from prefixes by performing the operation of position-by-position addition (modulo 3) with each element in $C_3(3, 3)$, where $C_3(3, 3) = [000, 111, 222]$.

prefixes $S_3(2)$	suffixes $S_3(2) \oplus 000$	prefixes $S_3(2)$	suffixes $S_3(2) \oplus 111$	prefixes $S_3(2)$	suffixes $S_3(2) \oplus 222$
00	000	00	111	00	222
01	010	01	121	01	202
02	020	02	101	02	212
10	100	10	211	10	022
11	110	11	221	11	002
12	120	12	201	12	012
20	200	20	011	20	122
21	210	21	021	21	102
22	220	22	001	22	112

For $n \leq 4$, a slight modification of the generation scheme (5) will increase the size of codes considerably, and they are shown in the following examples for $n = 3$ and 4.

Example 2:

$n = 3$ and $k = 3$ prefixes	suffixes	$n = 4$ and $k = 3$ prefixes	suffixes
0	$0 \oplus 00$	00	$00 \oplus 00$
1	$1 \oplus 00$	01	$01 \oplus 00$
2	$2 \oplus 00$	02	$02 \oplus 00$
		10	$10 \oplus 00$
1	$1 \oplus 11$	11	$11 \oplus 00$
2	$2 \oplus 22$	12	$12 \oplus 00$
		20	$20 \oplus 00$
		21	$21 \oplus 00$
		22	$22 \oplus 00$
		01	$01 \oplus 11$
		10	$10 \oplus 11$
		11	$11 \oplus 11$
		02	$02 \oplus 22$
		20	$20 \oplus 22$
		22	$22 \oplus 22$

[6] The author was introduced to this problem in correspondence with B. B. Barrow, SHAPE, Air Tech. Center, The Hague, The Netherlands.

[7] W. H. Kim and C. V. Freiman, "Single error correcting codes for asymmetric binary channels," IRE Trans. on Information Theory, vol. IT-5, pp. 62–66; June, 1959.

[8] W. H. Kim and C. V. Freiman, "Multi-error correcting codes for an asymmetric binary channel," *Trans. 1959 Internatl. Symp. on Circuit and Information Theory*, pp. 71–78; June, 1959.

[9] E. N. Gilbert, "Gray codes and paths on the n-cube," *Bell Sys. Tech. J.*, vol. 37, pp. 815–826; May, 1958.

[10] $a_k(n, 3)$ denotes the size (number of elements) of the largest single (± 1)-error correcting code in $S_k(n)$ and $b_k(n, 3)$ the size of the largest single (± 1)-error correcting group code in $S_k(n)$. (See Lee.[3])

The number of elements in a (-1)-error correcting code is denoted by $h_k(n, 2)$ and is given

$$h_k(n, 2) = k^r b_k(n - r, 3)$$

or

$$k^r a_k(n - r, 3) \quad \text{for} \quad n \geq 5 \quad \text{and} \quad k \geq 2. \tag{6}$$

For comparison of an asymmetric code and a symmetric *group* code $C_k(n, 3)$, the size of each code is shown below in Table I for $k = 3$ and n up to 14.

TABLE I
THE SIZE OF ASYMMETRIC AND SYMMETRIC TERNARY CODES

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$b_3(n, 3)$	1	3	3^2	3^2	3^3	3^4	3^5	3^6	3^7	3^8	3^9	3^{10}	3^{10}
$h_3(n, 2)$	3	5	15	3^3	3^4	3^5	3^6	3^6	3^7	3^8	3^9	3^{10}	3^{11}

From Theorems 1 and 2 in Lee's paper[3] and (6), Lemma 1 immediately follows.

Lemma 1: let k be an odd prime. If $n + 1$ is not a power of k,

$$kh_k(n, 2) = h_k(n + 1, 2) \quad \text{for} \quad n \geq 5.$$

If $n + 1$ is the power of k, then any asymmetric code with $h_k(n, 2)$ words is a close-packed code,[3,9] and

$$h_k(n, 2) = h_k(n + 1, 2) \quad \text{for} \quad n \geq 5.$$

Remark 1: if values for k and n are so chosen that

$$b_k(n, 3) = k^r b_k(n - r, 3)$$

or

$$k^r = \frac{b_k(n, 3)}{b_k(n - 3, 3)}$$

then the sizes of both symmetric group code $C_k(n, 3)$ and the asymmetric code designed by the proposed scheme are the same.

Multi-Small-Error Asymmetric Correction

For an e-tuple asymmetric correcting ternary code, the generation scheme (5) is directly applicable with some modifications: divide a given length of a word into $(e + 1)$ letters and use an auxiliary code $C_3(r, 2e + 1)$ for $n = (e + 1)r + c$, where $c = 0, 1, 2, \cdots, e$. This generation scheme is a direct extension of the method used for an e-tuple 1-error correcting code for a binary asymmetric channel.[7]

When $k > 3$, the minimum distance requirement (3c) may enable us to generate a code in a very simple manner. For an example, let $k = 5$ and $n = 3$; then a double (-1)-error [or $(+1)$-error] correcting code is shown in Example 3.

Example 3: $n = 3$, and $k = 5$. Then the the first group of words for a double (-1)-error-correcting code is generated as:

prefixes	stems[7]	suffixes
0	$0 \oplus 0$	$0 \oplus 0$
1	$1 \oplus 0$	$1 \oplus 0$
2	$2 \oplus 0$	$2 \oplus 0$
3	$3 \oplus 0$	$3 \oplus 0$
4	$4 \oplus 0$	$4 \oplus 0$

Next we will add 2 to each letter of the words in the first group, and we have

$0 \oplus 2$	0	0	0	$0 \oplus 2$	0	0	0	$0 \oplus 2$
$1 \oplus 2$	1	1	1	$1 \oplus 2$	1	1	1	$1 \oplus 2$
$2 \oplus 2$	2	2	2	$2 \oplus 2$	2	2	2	$2 \oplus 2$
$3 \oplus 2$	3	3	3	$3 \oplus 2$	3	3	3	$3 \oplus 2$
$4 \oplus 2$	4	4	4	$4 \oplus 2$	4	4	4	$4 \oplus 2$

As shown in Example 3, a sufficient condition for a pair of words s and t in order to correct all single and double (-1)-errors when $k > 3$ and $n \geq 2$, is that $\rho(s_i, t_i) \geq 2$ for $i = 1, 2, \cdots, n$. Therefore, the condition that $\rho(s_i, t_i) \geq e$ for $k \geq 2e \geq 4$ and $n \geq e$, is sufficient for e-tuple (-1)-error correction.

Remark 2: The diophantine equation for a close-packed, asymmetric double error correcting code is given, for $k \geq 3$,

$$n + \frac{n!}{(n - 2)!2!} + 1 = k^x. \tag{7}$$

Some solutions of (7) are found to be, for $k = 4$,

$$\begin{array}{lll} n = 2 & n = 5 & n = 90 \\ k = 4 & k = 4 & k = 4 \\ x = 1 & x = 2 & x = 6\ . \end{array} \tag{8}$$

The first set of solutions in (8) is shown in Example 4. This existence of 4-valued, closed-packed codes may be a significant feature for asymmetric correcting codes since there is no 4-valued, closed-packed, double error correcting code for a symmetric nonbinary channel.[3]

Example 4: A 4-valued, close-packed double (-1)-error correcting code for $n = 2$ and $k = 4$ is shown.

Code Words	00	12	20	32
Received sequences assigned to the above code word	03	02	10	22
	30	11	23	31
	33	01	13	21

APPENDIX

Let $x_i, x_j \in S_k(r)$ and $c_i, c_j \in C_k(n - r, 3)$; then four distinct letters in a (-1)-error correcting code are generated by performing the operation of (5) as:

$$s = (x_i, x_i \oplus c_i), \quad t = (x_j, x_j \oplus c_j)$$

$$u = (x_i, x_i \oplus c_j), \quad v = (x_j, x_j \oplus c_i)$$

for $i \neq j$.

To check the weight-distance requirement for each pair of letters, we will use the following lemmas without proof since they may be checked readily.

Lemma 2: If $\Delta W(x_i, x_j) = 0$ for $i \neq j$, then $\rho(x_i, x_j) \geq 2$ for $k \geq 2$ and $r \geq 2$.

Lemma 3: Let $\rho(x_i, x_j) = d_1$ and $\rho(x_i \oplus c_i, x_j \oplus c_j) = d_2$. Then,

$$\rho(c_i, c_j) = |d_1 - d_2| + 2a$$

where

$$d_1 = 0, 1, \cdots, \frac{r(k - 1)}{2},$$

$$d_2 = 0, 1, \cdots, \frac{r(k - 1)}{2}$$

and

$$a = 0, 1, \cdots, \text{Min}\left\{d_1, d_2, \frac{r(k - 1)}{2} - d_1, \frac{r(k - 1)}{2} - d_2.\right\}$$

From Lemma 2, it is clear that

$$\rho(s, t) \geq 4 \quad \text{for} \quad \Delta W(x_i, x_i) = 0$$

$$\Delta W(s, t) \geq 2 \quad \text{for} \quad \Delta W(x_i, x_j) = 1$$

which satisfy the condition of (4). The same is true for $\rho(u, v)$.

Since c_i and c_j were chosen such that $\rho(c_i, c_j) \geq 3$, it is obvious that

$$\rho(s, u) \geq 3 \quad \text{and} \quad \rho(t, v) \geq 3.$$

The remaining combinations to investigate are $\rho(s, v)$ and $\rho(t, u)$. A proof that the pair of letters s and v satisfy the weight-distance requirement for single (-1)-error correction will be given. For the letters t and u, a proof may be obtained in a similar manner.

Since an auxiliary code $C_k(n - r, 3)$ is used to generate suffixes, $\rho(c_i, c_j) \geq 3$ and $c_i, c_j \in C_k(n - r, 3)$. Then, we have from Lemma 3

$$\rho(c_i, c_j) = |d_1 - d_2| + 2a \geq 3 \tag{9}$$

and $\rho(s, v) = \rho(x_i, x_j) + \mu(x_i + c_i, x_j + c_j) = d_1 + d_2$. Now we are going to prove that

$$d_1 + d_2 \geq 3 \quad \text{for} \quad d_1 \geq 1, \quad \text{with} \quad i \neq j. \tag{10}$$

From (9), if $d_2 = 0$, then $a = 0$ and $d_1 \geq 3$, consequently $\rho(s, v) = d_1 \geq 3$. If $d_2 = 1$, then $a = 1$ and $d_1 = 2$, consequently $\rho(s, v) = d_1 + d_2 \geq 3$. For $d_2 \geq 2$, $\rho(s, v) = d_1 + d_2 \geq 3$, since $d_1 \geq 1$. Hence, a code generated by the proposed scheme satisfies the weight-distance requirement for single asymmetric correction.

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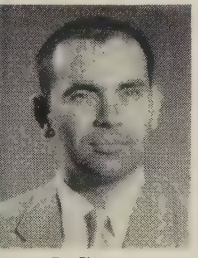
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Norman M. Abramson was born on April 1, 1932, in Boston, Mass. He received the B.A. degree in physics from Harvard College, Cambridge, Mass., in 1953, the M.A. degree in physics from the University of California at Los Angeles in 1955, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, Calif., in 1958.

He attended U.C.L.A. under the Hughes Cooperative Fellowship Plan, working part-time as a research engineer in the Systems Analysis Laboratory of Hughes Aircraft Company. In 1955 he entered the Electrical Engineering Department at Stanford under the Consolidated Engineering Fellowship; in 1956 he was appointed research assistant at Stanford Electronics Laboratory. In 1958 he was appointed acting assistant professor, and in 1959, assistant professor, of electrical engineering at Stanford. Since June, 1957, he has acted as consultant to the IBM Research Laboratories at San Jose, Calif. in the applications of statistical methods to engineering problems.

Dr. Abramson is the author of "Statistical Decision in Electronic Systems" to be published by McGraw-Hill Book Co., Inc.

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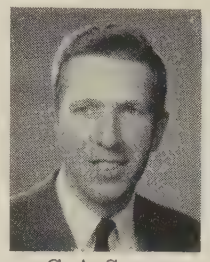


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Jack Capon (S '54—M '56) was born in New York, N. Y., on April 28, 1932. He received the B.E.E. degree from the College of the City of New York in 1953, the M.S.E.E. degree from the Massachusetts Institute of Technology, Cambridge, in 1955, and the Ph.D. degree in electrical engineering from Columbia University, New York, N. Y., in 1959.

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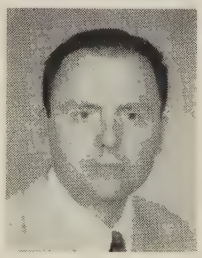
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From 1944 to 1946, he was employed in the research department of the Stromberg-Carlson Company, Rochester, N. Y. In 1947 and 1948, he was a recipient of an industrial electronics fellowship at M.I.T., and from 1948 to 1951 he served there as research assistant in the Research Laboratory of Electronics and instructor in the Electrical Engineering Department. In 1951, he was in England participating in transatlantic FM studies conducted by the Research Laboratory of Electronics. In 1952 he joined the M.I.T. Lincoln Laboratory, where he was engaged principally in radar and long-range communication research. In 1957, he joined the Information Studies Section of the General Electric Research Laboratory in Schenectady, N. Y., where he is presently engaged in communications research and anti-submarine warfare studies.

Dr. Stutt is a member of Sigma Tau, Pi Mu Epsilon, and Sigma Xi.

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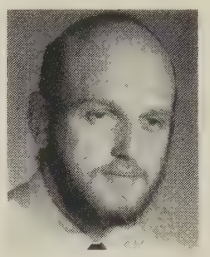
J. A. McFadden

James A. McFadden was born in San Juan, Puerto Rico, on December 11, 1924. He received the B.S.E. degree in mathematics, the B.S.E.E. degree, and the M.S. and Ph.D. degrees in physics in the years 1945, 1946, 1947, and 1951, respectively, all from the University of Michigan, Ann Arbor.

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Dr. McFadden is a member of the American Physical Society, the ASEE, Tau Beta Pi, Phi Kappa Phi, and Sigma Xi.

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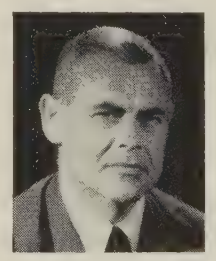


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Michael B. Marcus was born in Brooklyn, N. Y., on March 5, 1936. He received the B.S. degree in electrical engineering from Princeton University, Princeton, N. J., in 1957, and, as a National Science Foundation Fellow, he received the M.S. degree in mathematics from the Massachusetts Institute of Technology, Cambridge, in 1958.

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Nelson Wax was born in Philadelphia, Pa., on April 2, 1917. He received the B.S.E.E. and M.S.E.E. degrees in 1937 and 1938, respectively, both from the University of Pennsylvania, Philadelphia, and the Ph.D. degree from The Ohio State University, Columbus, in 1942. From 1942–1948, he was a member of the technical staff at Bell Telephone Laboratories, Inc., where he was engaged in the design and development of microwave magnetrons. He has been at the University of Illinois, Urbana, since 1948 except for the year 1954–1955, when he was on leave as a Guggenheim Fellow at the Research Laboratory of Electronics, Chalmers Institute of Technology, Gothenburg, Sweden. He is Professor of electrical engineering, and Research Professor at the Coordinated Science Laboratory, University of Illinois.

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All technical manuscripts and editorial correspondence should be addressed to George A. Deschamps, University of Illinois, Urbana, Ill. Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to Laurin G. Fischer, ITT Laboratories, 492 River Road, Nutley 10, N. J.

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